# MATH 150 Worksheet Keys Exam 4

## Worksheet for Section 1

1. Find $y'$ and $y''$ if

$$y = \frac{ln\ x}{x^2}$$

$$y' = \frac{x^2 \frac{d}{dx}(ln\ x) - [ln\ x\ \frac{d}{dx}(x^2)]}{(x^2)^2} = \frac{x - 2x(ln\ x)}{x^4} = \frac{1 - 2ln\ x}{x^3}$$

$$y'' = \frac{x^3 \frac{d}{dx}(1 - 2ln\ x) - [(1 - 2ln\ x)\frac{d}{dx}(x^3)]}{(x^3)^2} = \ldots \frac{6ln\ x - 5}{x^4}$$

2. Use logarithmic differentiation to find $y'$ if $y = (sin\ x)^x$

$$ln\ y = x\ ln(sin\ x) \iff \frac{y'}{y} = x\ \frac{1}{sin\ x}\ cos\ x + ln(sin\ x)(1) \iff$$

$$y' = y(x\ cot\ x + ln(sin\ x)) \iff y' = (sin\ x)^x(x\ cot\ x + ln(sin\ x))$$

3. Differentiate:

$$f(x) = ln\left(\frac{x}{x-1}\right)$$

using log properties $f(x) = ln\ x - ln\ \ (x-1) \implies$

$$f'(x) = \frac{1}{x} - \frac{1}{x-1}(1) = \frac{1}{x} - \frac{1}{x-1} = \frac{-1}{x(x-1)}$$

**Worksheet for Section 2**

1. A bacteria culture grows with a constant relative growth rate. After 2 hours there are 600 and after 8 hours there are 75,000.

   (a) Find the initial population.

   $$600 = y_0\, e^{2k} \text{ and } 75000 = y_0\, e^{8k} \implies \frac{75000}{600} = \frac{e^{8k}}{e^{2k}} = e^{6k} \implies k = \frac{ln\ 5}{2} \implies y_0 = 120$$

   (b) Find a formula for the population after $t$ hours.

   $$y(t) = 120\ e^{(ln\ 5)t/2} \text{ or } y = 120 \cdot 5^{t/2}$$

   (c) Find the number of bacteria after 5 hours.

   $$y(5) \approx 6700$$

   (d) Find the rate of growth after 5 hours.

   $$y(t) = 120 \cdot 5^{t/2} \text{ so } y'(t) = 60(ln\ 5) \cdot 5^{t/2} \text{ and } y'(5) \approx 5400/hour$$

   (e) When will the population be 200,000?

   $$200,000 = 120\ e^{(ln\ 5)t/2} \implies t \approx 9.2\ hours$$

2. After 3 days a sample of radon-222 decayed to 58% of its original amount.

   (a) What is its half-life?

   $$y(3) = A\ e^{3k} = .58A \implies k = \frac{ln\ .58}{3}$$

   $$\frac{1}{2}A = A\ e^{(ln\ .58)t/3} \implies t \approx 3.82\ days$$

   (b) How long until the sample decays to 10% of its original amount

   $$.1\ A = A\ e^{(ln\ .58)t/3} \implies t \approx 12.7\ days$$

**Worksheet for Section 3**

1. As a circular metal grate is being heated, its diameter changes at a rate of .01 $cm/min$. Find the rate at which the area of one side is changing when the diameter is 30 $cm$.

$$A = \pi \left(\frac{d}{2}\right)^2 \;=\; \frac{\pi}{4}d^2 \;\text{ so }\; \frac{dA}{dt} = \frac{\pi}{4}2d\frac{dd}{dt} = .15\pi\ cm^2/min$$

2. Suppose a spherical snowball is melting and the radius is decreasing at a constant rate, changing from 12 $in$ to 8 $in$ in 45 $min$. How fast was the volume changing when the radius was 10 $in$?

$$\frac{dr}{dt} = \frac{4}{45} = .089 \;\text{ and }\; V = \frac{4}{3}\pi r^3 \;\Longrightarrow\; \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} = 35.6\pi\ in^3/min$$

3. A ladder 20 $ft$ long leans against a vertical wall. If the bottom of the ladder slides away from the building at a rate of 2 $ft/sec$, at what rate is the angle between the ladder and the ground changing when the top of the ladder is 12 $ft$ above the ground?

$$\frac{dx}{dt} = 2 \;\text{ and }\; cos\ \theta = \frac{x}{20} \;\Longrightarrow\; -\,sin\ \theta\frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt} \;\Longrightarrow\; \frac{d\theta}{dt} = -1/6\ rad/sec$$

4. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its $x$-coordinate increases at a rate of 3 $cm/sec$. How fast is the distance from the particle to the origin changing at this instant?

$$D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (\sqrt{x})^2} = \sqrt{x^2 + x}$$

$$\frac{dD}{dt} = \frac{2x+1}{2\sqrt{x^2+x}}\frac{dx}{dt} \;\text{ since }\; \frac{dx}{dt} = 3 \text{ when } x = 4 \;\Longrightarrow\; \frac{dD}{dt} \approx 3.02\ cm/sec$$

5. Two sides of a triangle are 4 $m$ and 5 $m$ in length and the angle between them is increasing at a rate of .06 $rad/s$. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.

$$h = 4sin\ \theta \;\text{ so }\; A = \frac{1}{2}(5)(4sin\ \theta) = 10sin\ \theta$$

$$\frac{dA}{dt} = 10cos\ \theta\frac{d\theta}{dt} \;\Longrightarrow\; \frac{dA}{dt} = .3\ m^2/sec$$

6. At noon, ship A is 100 $km$ west of ship B. Ship A is sailing south at 35 $km/hr$ and ship B is sailing north at 25 $km/hr$. How fast is the distance between the ships changing at 4:00 P.M.?

$$\frac{dx}{dt} = 35 \;\text{ and }\; \frac{dy}{dt} = 25 \;\text{ and }\; z^2 = (x+y)^2 + (100)^2$$

$$\Longrightarrow\; 2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \;\text{ so }\; \frac{dz}{dt} = \frac{720}{13} \approx 55.4\ km/hr$$

**Worksheet for Section 4**

1. Find the linearization, $L(x)$, of $f(x) = ln\ x$ at $a = 1$.

$$f(x) = ln\ x \qquad f(1) = 0 \qquad f'(x) = \frac{1}{x} \qquad f'(1) = 1$$

$$\Longrightarrow \quad L(x) = f(a)+f'(a)(x-a) = f(1)+f'(1)(x-1) = 0+1(x-1) \quad \Longrightarrow \quad L(x) = x-1$$

2. Find and evaluate $dy$ if $y = 1/(x+1)$, $x = 1$, $dx = -0.01$.

$$dy = f'(x)dx \qquad f(x) = \frac{1}{x+1} \qquad f'(x) = \frac{-1}{(x+1)^2}$$

$$\Longrightarrow \quad dy = \frac{-dx}{(x+1)^2} \text{ and when } x = 1\ ,\ dx = -.01 \quad \Longrightarrow \quad dy = \frac{-(-.01)}{2^2} = \frac{.01}{4}$$

**Worksheet for Section 5**

1. Prove the following identity: $cosh\ x + sinh\ x = e^x$

$$cosh\ x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad sinh\ x = \frac{e^x - e^{-x}}{2}$$

$$\implies \quad cosh\ x + sinh\ x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{2e^x}{2} = e^x$$

2. Find the derivative of $y = sinh(cosh\ x)$

$$y' = cosh(cosh\ x)\frac{d}{dx}(cosh\ x)$$

$$= (cosh(cosh\ x))(sinh\ x)$$

3. Find the derivative of

$$y = e^{cosh\ 3x}$$

$$y' = e^{cosh\ 3x}\frac{d}{dx}(cosh\ 3x)$$

$$= e^{cosh\ 3x}(3)(sinh\ 3x)$$