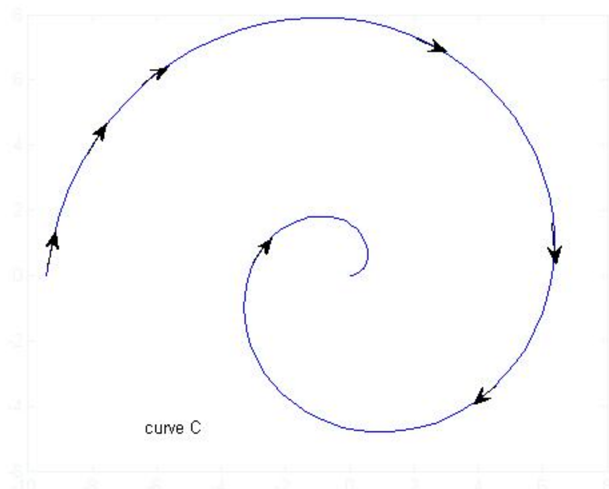
1 Curves Defined by Parametric Equations



As you can see, curve C can not be described as a function, $y = f(x)$. Why?

We can however describe C another way. As functions of time.

By letting x and y be functions of a third variable, say t, we would obtain

$x = f(t)$ and $y = g(t)$ with t acting as a parameter.

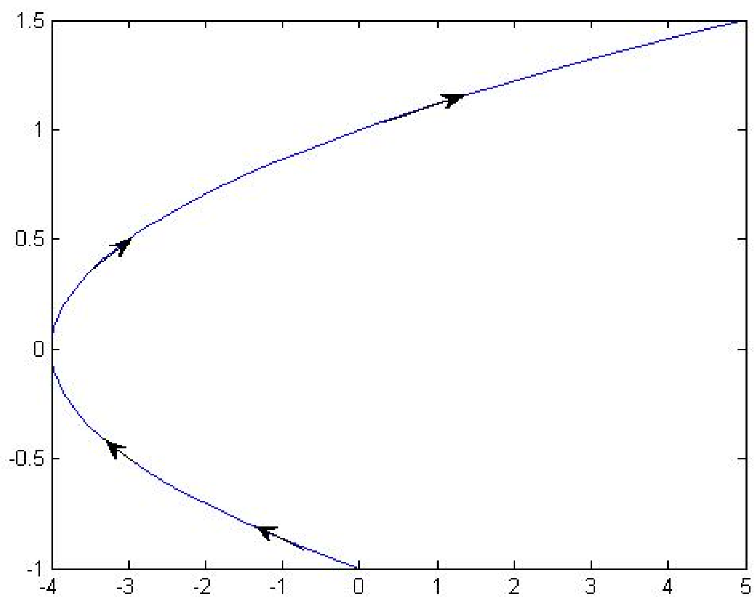
As t varies, the point $(x, y) = (f(t), g(t))$ also varies and traces out a curve C

ex 1 Sketch the curve described by the parametric equations

$$x = t^2 - 4 \qquad and \qquad y = \frac{t}{2} \qquad with \qquad -2 \leq t \leq 3$$

So, picking some values for t we get

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	-1/2	0	1/2	1	3/2



As you can see this curve has ***direction***

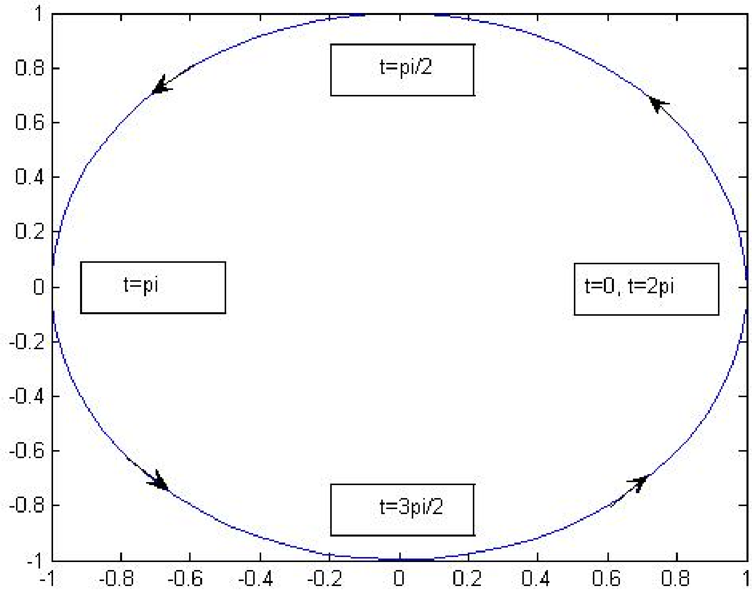
In general, a curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leq t \leq b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$

ex 2 Sketch the curve represented by the parametric equations

$$x = cos\ t \qquad y = sin\ t \qquad 0 \leq t \leq 2\pi$$



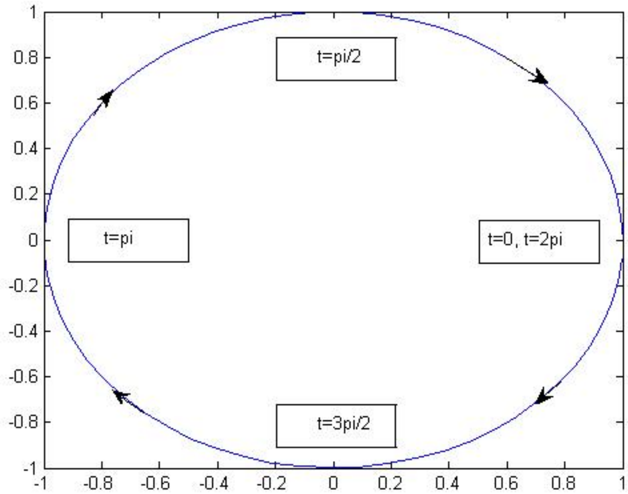
This appears to be a circle. We can confirm this by *eliminating the parameter*

$$x^2 + y^2 = cos^2\ t + sin^2\ t = 1$$

What would happen if we sketched

$$x = sin\ t \qquad y = cos\ t \qquad 0 \leq t \leq 2\pi$$

We get the same picture but now the direction is *clockwise*



What would happen if we sketched

$$x = sin\ 2t \qquad y = cos\ 2t \qquad 0 \le t \le 2\pi$$

We get a similar picture but now the curve goes around twice. You get the idea...

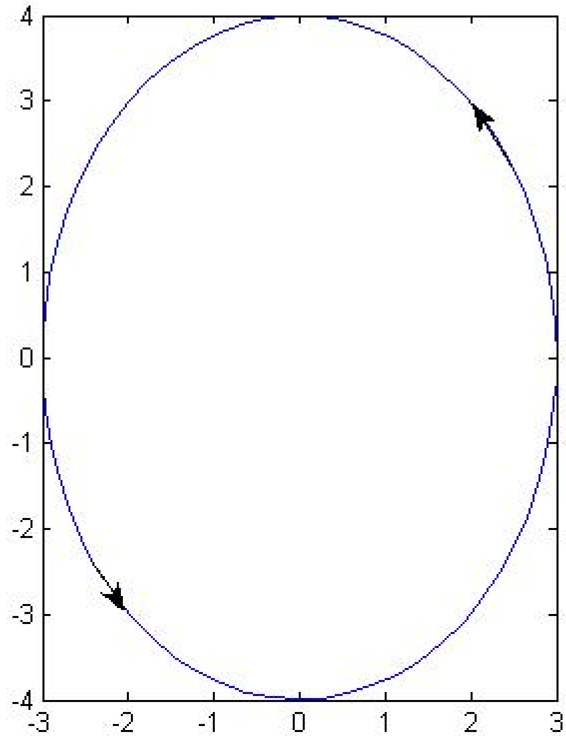
ex 3 Sketch the curve represented by the parametric equations

$$x = 3cos\ \theta \qquad y = 4sin\ \theta \qquad 0 \le \theta \le 2\pi$$

Let's eliminate the parameter, θ , to see what this should look like

$$cos^2\,\theta + sin^2\,\theta \ = \ 1 \quad \Longrightarrow \quad \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 \ = \ 1 \quad \Longrightarrow$$

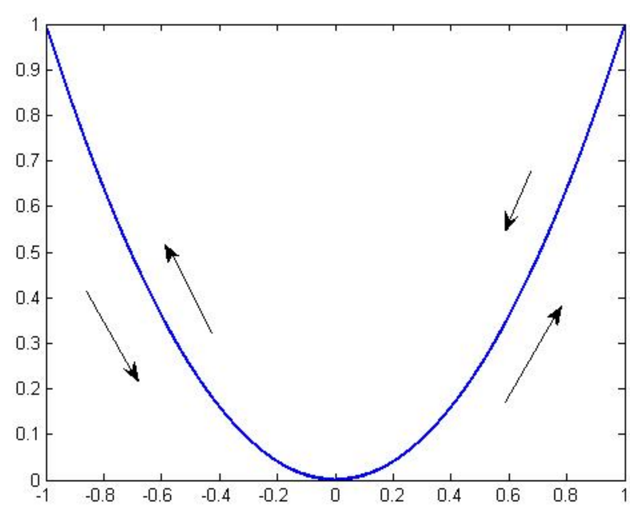
$$\frac{x^2}{9} + \frac{y^2}{16} \ = \ 1 \quad \Longrightarrow \quad an\ ellipse$$



ex 4 Sketch the curve represented by the parametric equations

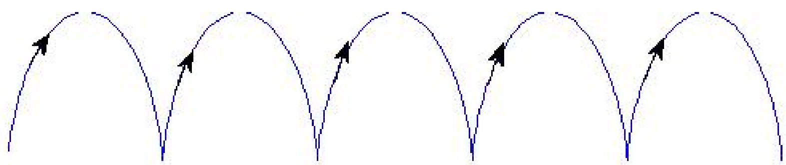
$$x = sin\ t \qquad y = sin^2\ t$$

Note that $y = x^2$ is a parabola, but $|sin\ t| \leq 1$ so $(x, y) = (sin\ t, sin^2\ t) \implies$ the object moves back and forth along the parabola from $(-1, 1)$ to $(1, 1)$ infinitely often.

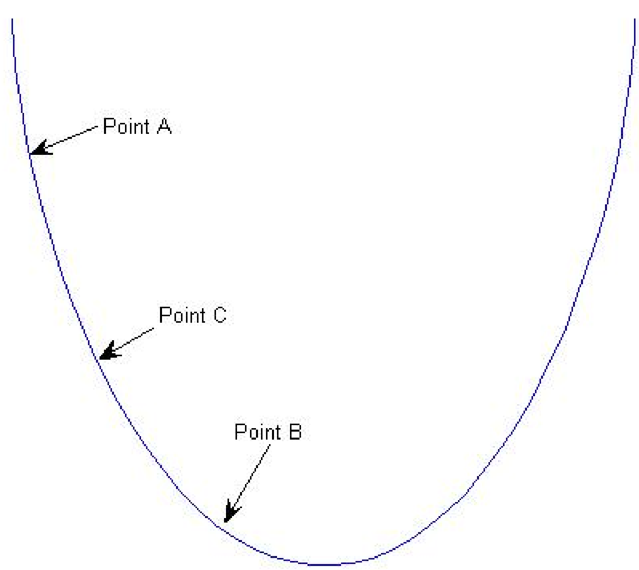


There is another interesting curve that can be traced out using parametric equations called a ***cycloid***

If you roll a sphere along a straight line the curve traced out by a point P on the surface of the sphere follows the following path



If we take one section or arch of the cycloid and invert it we get the following picture



If a ball is released from point C, do you think it will reach point B before a ball that is released from point A?

It turns out they will reach point B at exactly the same time. Why do you think this is?

Why do you think that a pendulum swings in the same arc as the inverted arch of a cycloid?

Worksheet for Section 1

1. Let $x = 1 + t$, $y = 5 - 2t$ and $-2 \leq t \leq 3$:

   (a) Sketch the curve and indicate direction as t increases.

   (b) Eliminate the parameter to find a Cartesian equation of the curve.

2. Describe the motion of the particle given by $x = 2 + cos\ t, \quad y = 3 + sin\ t, \quad 0 \leq t \leq 2\pi$

Homework for Section 1

1. Sketch $x = 1 + \sqrt{t}\ ,\quad y = t^2 - 4t\ ,\quad 0 \le t \le 5$
2. Sketch $x = t^2 - 2\ ,\quad y = 5 - 2t\ ,\quad -3 \le t \le 4$ and eliminate the parameter to find a Cartesian equation.
3. Eliminate the parameter and sketch $x = sin\ \theta\ ,\quad y = cos\ \theta\ ,\quad 0 \le \theta \le \pi$
4. Describe the motion of $x = 5sin\ t\ ,\quad y = 2cos\ t\ ,\quad -\pi \le t \le 5\pi$

2 Calculus with Parametrics

We will cover tangents, arc length and surface area with parametric curves.

2.1 Tangents

Since parametrics are defined in terms of both x and y, the derivative is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad provided \ \frac{dx}{dt} \neq 0$$

$$\implies \quad horizontal\ tangent\ when? \quad \frac{dy}{dt} = 0$$

$$\implies \quad vertical\ tangent\ when? \quad \frac{dx}{dt} = 0$$

The second derivative is slightly different so be careful!

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} \quad provided \ \frac{dx}{dt} \neq 0$$

That is, take the derivative of the first derivative with respect to t and then divide by dx/dt again.

ex 5 For the following curve, find the slope **and** concavity at $(2, 3)$

$$x = \sqrt{t} \quad y = \frac{1}{4}\left(t^2 - 4\right) \quad t \geq 0$$

$$(x, y) = (2, 3) \implies t = 4$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \ldots = t^{3/2} \implies slope = 8$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t^{3/2})}{\frac{dx}{dt}} = \ldots = 3t \implies CU\ at\ (2,3)$$

2.2 Arc Length

For parametric equations the formula is:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

provided that f' and g' are continuous on $[\alpha, \beta]$ and the curve is traversed exactly once as t goes from α to β

This is really just a technicality and you needn't concern yourself with it as any problem I provide will meet these conditions.

ex 6 Find the length around the unit circle where

$$x = cos\ t \quad y = sin\ t \quad and \quad 0 \le t \le 2\pi$$

$$dx/dt = -\ sin\ t \quad dy/dt = cos\ t \implies$$

$$L = \int_0^{2\pi} \sqrt{sin^2\ t + cos^2\ t}\ dt = \int_0^{2\pi} dt = 2\pi$$

as expected

2.3 Surface Area

Recall

$$\text{rotation about the } x\text{-axis}$$
$$S = \int_\alpha^\beta 2\pi y \ ds$$
$$\text{rotation about the } y\text{-axis}$$
$$S = \int_\alpha^\beta 2\pi x \ ds$$

$$but \ \ ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt \quad either \ way$$

ex 7 Find the surface area by rotating $x = 3t^2, \ \ y = 2t^3 \ \ 0 \le t \le 5$ about the y-axis

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (6t)^2 + (6t^2)^2 = 36t^2(1+t^2) \Longrightarrow$$

$$SA = \int_0^5 2\pi 3t^2 6t\sqrt{1+t^2} \ dt = 18\pi \int_0^5 t^2\sqrt{1+t^2}2t \ dt$$

$$and \ if \ u = 1+t^2, \ u-1 = t^2 \ so \ du = 2t \ dt \quad \Longrightarrow$$

$$18\pi \int_1^{26} (u-1)\sqrt{u} \ du = \ \dots \ = \frac{24}{5}\pi\left(949\sqrt{26}+1\right)$$

Worksheet for Section 2

1. Find an equation of the tangent to $x = t^4 + 1\ ,\ \ y = t^3 + t$ at the point corresponding to $t = -1$.

2. Find the length of the curve $x = e^t\, cos\, t,\ \ y = e^t\, sin\, t,\ \ 0 \le t \le \pi$

3. Find the area of the surface obtained by rotating $x = 3t - t^3,\ \ y = 3t^2,\ \ 0 \le t \le 1$
   about the $x - axis$.

Homework for Section 2

1. Find an equation of the tangent line to $x = t^4 + 1$, $y = t^3 + 4$ at $t = -1$

2. Find an equation of the tangent line to $x = e^{\sqrt{t}}$, $y = t - ln\ t^2$ at $t = 1$

3. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following as well as when the curves are CU.

   (a) $x = 4 + t^2$, $y = t^2 + t^3$

   (b) $x = t - e^t$, $y = t + e^{-t}$

4. SET UP ONLY the integral that represents the length of

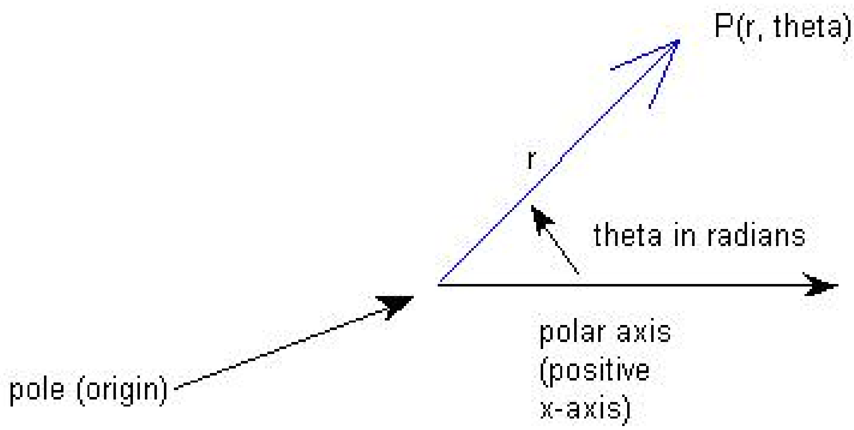
$$x = t - t^2 \ , \ \ y = 4/3t^{3/2} \ , \ \ 1 \leq t \leq 2$$

5. Find the exact length of $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$

6. Find the exact length of $x = e^t cos\ t$, $y = e^t sin\ t$, $0 \leq t \leq \pi$

7. Find the surface area by rotating $x = acos^3\,\theta$, $y = asin^3\,\theta$, $0 \leq \theta \leq \pi/2$ about the x-axis.

3 Polar Coordinates



If the angle is positive then it is measured counterclockwise

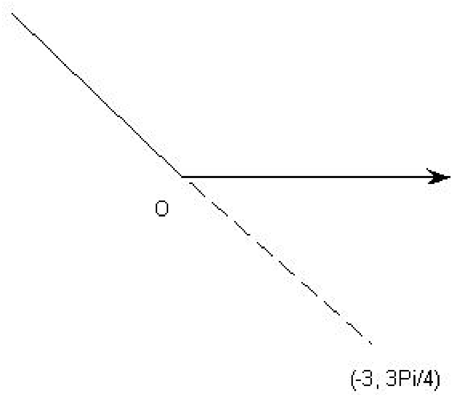
We will use the convention that if r is negative, the points $(-r, \theta)$ and (r, θ) lie on the same line through the origin and at the same distance, $\mid r \mid$ from the origin but on opposite sides.

$\Longrightarrow$ If $r > 0$ then (r, θ) is in the same quadrant as θ

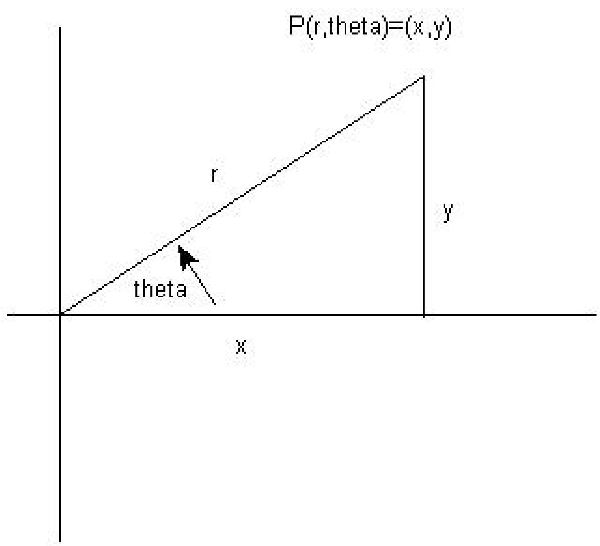
$\Longrightarrow$ If $r < 0$ then (r, θ) is in the quadrant on the opposite side

$$\textit{that is,} \qquad (-r, \theta) \; = \; (r, \theta + \pi)$$

ex 8 Plot $(-3, 3\pi/4)$



WHAT IS THE CONNECTION BETWEEN POLAR AND CARTESIAN COORDINATES?



Note that:

$$cos\ \theta\ =\ \frac{x}{r}\quad sin\ \theta\ =\ \frac{y}{r}\quad and\quad x^2+y^2=r^2$$

If Polar is known then Cartesian is $\Longrightarrow$ $\mathbf{x = r\ cos\ \theta\ \ and\ \ y = r\ sin\ \theta}$
If Cartesian is known then Polar is $\Longrightarrow$ $\mathbf{r^2 = x^2 + y^2\ \ and\ \ tan\ \theta\ =\ \frac{y}{x}}$

ex 9 Convert $(\sqrt{3}, \pi/6)$ to Cartesian coordinates

$$x = \sqrt{3}\ cos\ \pi/6\quad and\quad y = \sqrt{3}\ sin\ \pi/6\ \ \Longrightarrow\ \ (x, y)\ =\ \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$$

ex 10 Convert $(-1, 1)$ to Polar coordinates

$$tan\ \theta\ =\ \frac{y}{x}\ =\ -1\ \Longrightarrow\ \theta\ =\ \frac{3\pi}{4}$$

Since θ is in the same quadrant as $(-1, 1)$ use $r > 0$ where $r = \sqrt{x^2+y^2}\ =\ \sqrt{2}$

So, **ONE** set of polar coordinates is

$$\left(\sqrt{2}, \frac{3\pi}{4}\right)$$

We could also use

$$\left(-\sqrt{2}, \frac{7\pi}{4}\right)$$

Polar Graphs

One way is to convert to Cartesian

ex 11 Sketch $r = 2\ cos\ \theta$

So

$r^2 = 2rcos\ \theta \implies x^2 + y^2 = 2x\ \ since\ x^2 + y^2 = r^2\ and\ rcos\ \theta = x$

$\implies x^2 - 2x + y^2 = 0 \implies (x-1)^2 + y^2 = 1\ if\ you\ complete\ the\ square$

This is then a circle whose radius is 1 and whose center is $(1, 0)$

We can also use *parametric equations*

If you want to sketch $r = f(\theta)$ write:

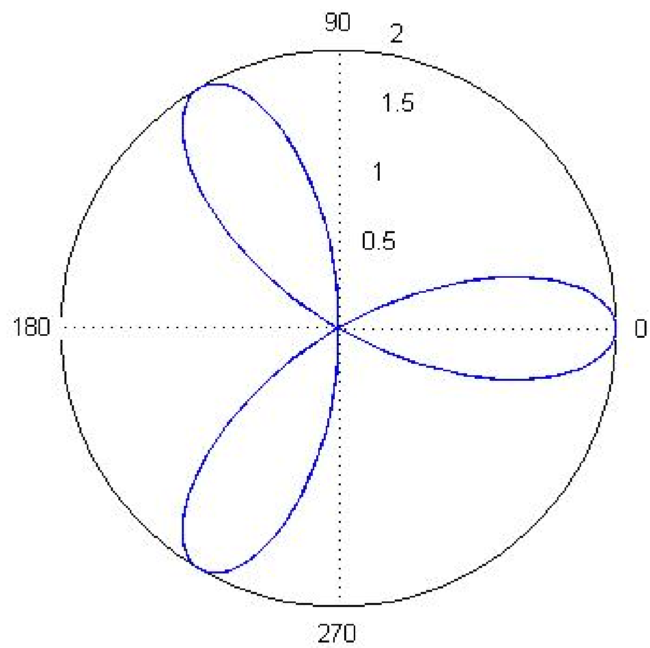
$$x = f(\theta)\ cos\ \theta \qquad and \qquad y = f(\theta)\ sin\ \theta$$

ex 12 Sketch $r = 2\ cos\ 3\theta$

Use the parametric equations:

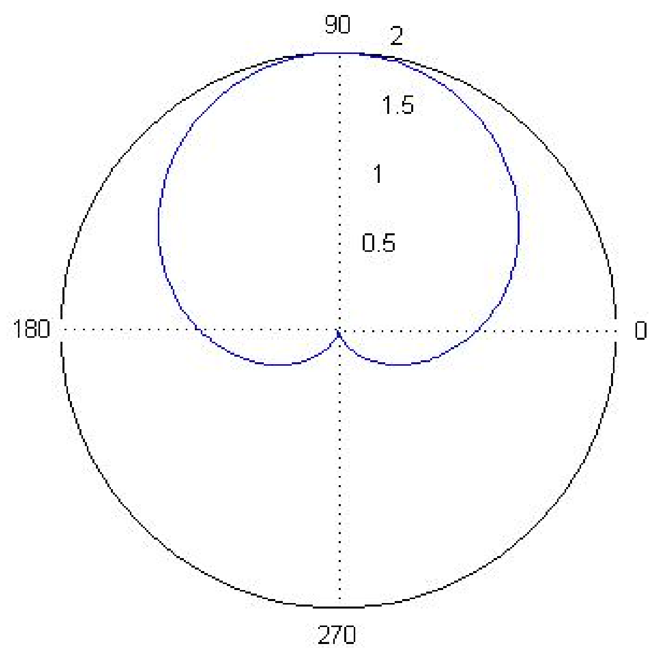
$$x = 2\ cos\ 3\theta\ cos\ \theta \qquad y = 2\ cos\ 3\theta\ sin\ \theta$$

You get the following *rose curve*



r = 2 cos(3 t)

Another popular polar shape is the *cardiod.* Here is an example of $r = 1 + sin\ \theta$



r = 1+sin(t)

Tangent Lines

Recall

$$x = f(\theta)\ cos\ \theta \qquad and \qquad y = f(\theta)\ sin\ \theta$$

So we will need the PRODUCT RULE!

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\ sin\ \theta + r\ cos\ \theta}{\frac{dr}{d\theta}\ cos\ \theta - r\ sin\ \theta}$$

Horizontal tangents occur when

$$\frac{dy}{d\theta} = 0 \ \ and \ \ \frac{dx}{d\theta} \neq 0$$

Vertical tangents occur when

$$\frac{dx}{d\theta} = 0 \ \ and \ \ \frac{dy}{d\theta} \neq 0$$

ex 13 Find the vertical and horizontal tangent lines of $r = sin\ \theta$

So

$$x = sin\ \theta\ cos\ \theta$$

$$y = sin\ \theta\ sin\ \theta \ = \ sin^2\ \theta$$

$$\frac{dx}{d\theta} \ = \ cos^2\ \theta - sin^2\ \theta \ = \ cos\ 2\theta \ = \ 0 \quad \Longrightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\frac{dy}{d\theta} = 2\ cos\ \theta\ sin\ \theta \ = \ sin\ 2\theta \ = \ 0 \quad \Longrightarrow \quad \theta = 0, \frac{\pi}{2}$$

Thus we have,

$$vertical\ tangents\ at \quad \left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)\ and\ \left(\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$$

$$horizontal\ tangents\ at \quad (0,0)\ and\ \left(1, \frac{\pi}{2}\right)$$

Worksheet for Section 3

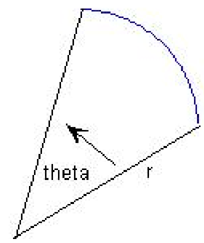
1. Identify the curve by finding a Cartesian equation for $r = 2\ sin\ \theta + 2\ cos\ \theta$
2. Find a polar equation represented by the Cartesian equation $x^2 + y^2 = 9$
3. Find the points on the curve $r = e^{\theta}$ where the tangent line is horizontal or vertical.

Homework for Section 3

1. Sketch the following region: $5\pi/3 \leq \theta \leq 7\pi/3$ for $2 < r < 3$
2. Identify by finding a Cartesian equation for $r = 2$
3. Identify by finding a Cartesian equation for $r = 3\ sin\ \theta$
4. Find a polar equation for the following:
   (a) $x = 3$
   (b) $x^2 + y^2 = 2cx$
5. Sketch the following:
   (a) $r = sin\ \theta$
   (b) $r = 2(1 - sin\ \theta), \quad \theta \geq 0$
   (c) $r = \theta, \quad \theta \geq 0$
6. Find the slope of the tangent line to $r = 2\ sin\ \theta$ at $\theta = \pi/6$
7. Find the slope of the tangent line to $r = 1/\theta$ at $\theta = \pi$
8. Find the slope of the tangent line to $r = cos\ 2\theta$ at $\theta = \pi/4$
9. Find the points on $r = 3\ cos\ \theta$ where the tangent line is horizontal or vertical.

4 Areas and Lengths in Polar Coordinates

Area in polar coordinates means utilizing sectors of a circle The entire



circle has area πr^2.
So a sector has area what?

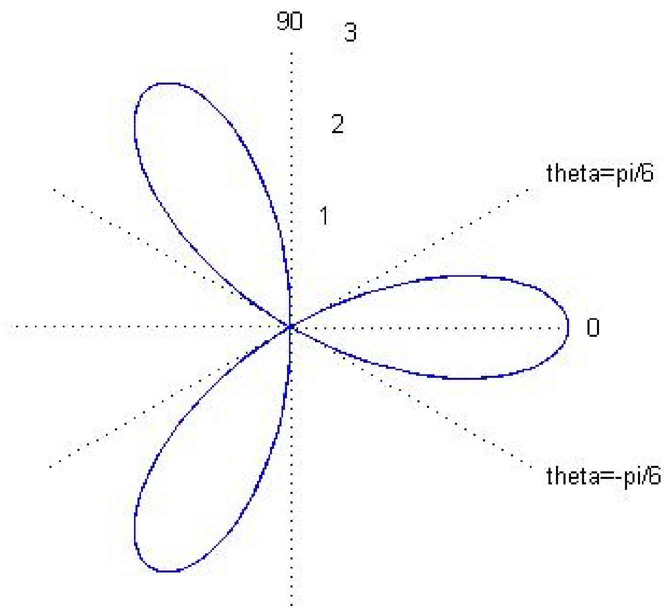
$$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}r^2\theta$$

$$\Longrightarrow \quad A_i = \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta \quad \Longrightarrow \quad A \approx \sum_{i=1}^{n} \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta$$

Taking a limit we obtain

$$A = \int_a^b \frac{1}{2}\left[f(\theta)\right]^2 d\theta \quad or \quad \int_a^b \frac{1}{2}r^2\, d\theta$$

ex 14 Find the area of one petal of the rose curve $r = 3\ cos\ 3\theta$



r = 3 cos(3 t)

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} r^2\, d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (3\ cos\ 3\theta)^2\, d\theta = \frac{9}{2}\int_{-\pi/6}^{\pi/6} \frac{1 + cos\ 6\theta}{2}\, d\theta$$

$$= \frac{9}{4}\left[\theta + \frac{sin\ 6\theta}{6}\right]_{-\pi/6}^{\pi/6} = \frac{3\pi}{4}$$

ex 15 Find the area common to the two regions bounded by $r = -6\ cos\ \theta$ and $r = 2 - 2\ cos\ \theta$

For this example as well as the next one I will need to demonstrate graphically on the board in class. For now, be aware that finding the points of intersection can be a little tricky in polar coordinates.

We will eventually arrive at the area being:

$$\frac{A}{2} \;=\; \frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6cos\;\theta)^2\, d\theta \;+\; \frac{1}{2}\int_{2\pi/3}^{\pi} (2 - 2cos\;\theta)^2\, d\theta \;=\; \ldots \;=\; 5\pi$$

Arc Length

for

$$x = f(\theta)\; cos\;\theta \quad and \quad y = f(\theta)\; sin\;\theta$$

$$\Longrightarrow \quad \frac{dx}{d\theta} \;=\; \frac{dr}{d\theta}\; cos\;\theta - r\; sin\;\theta \quad and \quad \frac{dy}{d\theta} \;=\; \frac{dr}{d\theta}\; sin\;\theta + r\; cos\;\theta$$

So

$$\left(\frac{dx}{d\theta}\right)^2 \;+\; \left(\frac{dy}{d\theta}\right)^2 \;=\; \ldots \;=\; \left(\frac{dr}{d\theta}\right)^2 \;+\; r^2$$

Therefore the arc length is

$$L \;=\; \int_a^b \sqrt{r^2 \;+\; \left(\frac{dr}{d\theta}\right)^2}\; d\theta \quad with \quad \frac{dr}{d\theta} \quad being\; continuous$$

ex 16 Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid $r = 2 - 2\; cos\;\theta$

So

$$L \;=\; \int_0^{2\pi} \sqrt{(2 - 2\; cos\;\theta)^2 \;+\; (2\; sin\;\theta)^2}\; d\theta \;=\; 2\sqrt{2}\int_0^{2\pi} \sqrt{1 - cos\;\theta}\; d\theta$$

$$=\; 2\sqrt{2}\int_0^{2\pi} \sqrt{2\; sin^2\,\frac{\theta}{2}}\; d\theta \;=\; 4\int_0^{2\pi} sin\,\frac{\theta}{2}\; d\theta \;=\; 16$$

Worksheet for Section 4

1. Sketch the curve $r = 3\ cos\ \theta$ and find the area in encloses.

2. Find the area enclosed by one loop of $r = 3\ cos\ 5\theta$

3. Find the length of the curve $r = e^{2\theta}$, $0 \le \theta \le 2\pi$

Homework for Section 4

1. Find the area of the region bounded by $r = sin\ \theta$ on $\pi/3 \leq \theta \leq 2\pi/3$
2. Sketch $r = 3\ cos\ \theta$ and find the area it encloses.
3. Find the area enclosed by ONE loop of $r = sin\ 2\theta$
4. Find the area that lies inside $r = 3\ cos\ \theta$ and outside $r = 1 + cos\ \theta$
5. Find the area that lies in both $r = \sqrt{3}\ cos\ \theta$ and $r = sin\ \theta$
6. Find the exact length of the polar curve $r = 3\ sin\ \theta$ from $0 \leq \theta \leq \pi/3$