MATH 150 Worksheet Keys Exam 5

MATH 150 Section 1 Worksheet Key Exam 5

1. Sketch the graph of $f(x) = x^2$, $0 < x \leq 2$ and use your sketch to find the absolute and local max(s) and min(s).

absolute max at $(2, 4)$

absolute min at (none)

local max at (none)

local min at (none)

2. Find the critical numbers of $g(x) = x^3 + x^2 + x$

$$g'(x) = 3x^2 + 2x + 1 \quad \text{and} \quad g'(x) = 0 \quad \Longleftrightarrow \quad x = \frac{-2 \pm \sqrt{4 - 12}}{6}$$

since neither is a real number $\Longrightarrow$ no critical numbers

1. Verify that $f(x) = x^3 + x - 1$ on the interval $[0, 2]$ satisfies the hypothesis of the MVT. Then find all numbers c that satisfy the conclusion of the MVT.

f is a polynomial so continuous and differentiable on $[0, 2]$ and $(0, 2)$

by the MVT $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} \iff 3c^2 + 1 = 5 \implies c = \pm\dfrac{2}{\sqrt{3}}$

however, only $+\dfrac{2}{\sqrt{3}} \in (0, 2)$

2. Suppose that $3 \leq f'(x) \leq 5$ for all values of x. Show that $18 \leq f(8) - f(2) \leq 30$.

if $3 \leq f'(x) \leq 5$ then by the MVT

$f(8) - f(2) = f'(c)(8 - 2)$ for some $c \in [2, 8] \implies f(8) - f(2) = 6 \cdot f'(c)$

so it follows that $6 \cdot 3 \leq 6f'(c) \leq 6 \cdot 5 \implies 18 \leq f(8) - f(2) \leq 30$

MATH 150 **Section 3 Worksheet Key** **Exam 5**

1. Find the intervals the function increases and decreases.
2. Find the local maximum and minimum values.
3. Find the intervals of concavity and the inflection points.

(a) $h(t) = 3t^5 - 5t^3 + 3$

increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$

$x = -1$ is a local max $\quad x = 1$ is a local min

CU on $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$

CD on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(0, \frac{1}{\sqrt{2}}\right)$

inflection points at $x = 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

(b) $g(x) = (x^2 - 1)^3$

increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$

$x = 0$ is a local min

CU on $(-\infty, -1)$, $\left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ and $(1, \infty)$

CD on $\left(-1, -\frac{1}{\sqrt{5}}\right)$ and $\left(\frac{1}{\sqrt{5}}, 1\right)$

inflection points at $x = 1, -1, \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}$

1. A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

Letting each side of the base be $3-2x$ and the height x we get the following

$$V = 9x - 12x^2 + 4x^3 \implies V' = 9 - 24x + 12x^2 \implies V' = 0 \iff x = \frac{3}{2}, \frac{1}{2}$$

the largest volume is then 2

2. Show that of all the rectangles with a given area, the one with the smallest perimeter is a square.

Letting $A = l \cdot w$ we want to maximize $P = 2l + 2w \implies P = 2\frac{A}{w} + 2w$

$$P' = \frac{2(w^2 - A)}{w^2} \implies \text{the critical number is } w = \sqrt{A} \implies \text{sides are } \sqrt{A} \text{ and } \sqrt{A}$$

therefore it's a square

3. Show that of all the rectangles with a given perimeter, the one with the greatest area is a square.

Since $P = 2x+2y \implies y = \dfrac{P-2x}{2}$ we want to maximize $A = x\left(\dfrac{P-2x}{2}\right) = \dfrac{1}{2}Px - x^2$

$A' = \dfrac{1}{2}P - 2x$ so $A' = 0 \iff x = \dfrac{P}{4} \implies$ sides are $\dfrac{P}{4}$ and $\dfrac{P}{4}$ therefore it's a square

4. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

If the base of the triangle is L, inscribe a rectangle such that the height of the rectangle is y and one half the base of the rectangle is x. Since the height of the triangle is $\dfrac{\sqrt{3}}{2}L$ we get the following

$$\text{from similar triangles } \frac{\frac{\sqrt{3}}{2}L - y}{x} = \frac{\frac{\sqrt{3}}{2}L}{\frac{L}{2}} \implies y = \frac{\sqrt{3}}{2}(L - 2x)$$

$$A_{\text{rectangle}} = 2xy = \sqrt{3}Lx - 2\sqrt{3}x^2 \text{ so } A' = \sqrt{3}L - 4\sqrt{3}x = 0 \iff x = \frac{L}{4}$$

thus the dimensions are $\dfrac{L}{2}$ and $\dfrac{\sqrt{3}}{4}L$

5. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

Inscribe a cylinder in a sphere and let the radius r go from the center of the sphere to a corner of the cylinder. Then let the radius of the cylinder be y and half the height be x.

The cylinder then has volume $V = \pi y^2 2x$ and since $x^2 + y^2 = r^2 \implies$

$V = \pi(r^2 - x^2)2x = 2\pi(r^2 x - x^3)$ therefore $V' = 2\pi(r^2 - 3x^2) = 0 \iff x = \dfrac{r}{\sqrt{3}}$

so the largest volume is $V\left(\dfrac{r}{\sqrt{3}}\right) = \dfrac{4\pi r^3}{3\sqrt{3}}$

MATH 150 Section 4 Worksheet (B) Key Exam 5

1. A cylindrical can without a top is made to contain a volume V of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

$$V = \pi r^2 h \text{ and } SA = \pi r^2 + 2\pi r h \text{ so } SA = \pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) \implies$$

$$SA = \pi r^2 + \frac{2V}{r} \text{ therefore } (SA)' = 2\pi r - \frac{2V}{r^2} \text{ and } (SA)' = 0 \iff r = \sqrt[3]{\frac{V}{\pi}}$$

$$\text{so } r = \sqrt[3]{\frac{V}{\pi}} \text{ and } h = \frac{V}{\pi r^2} = \sqrt[3]{\frac{V}{\pi}} \text{ as well}$$

2. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into a circle. How should the wire be cut so that the total area enclosed is:

   (a) a maximum?

   (b) a minimum?

take a 10m length of wire and cut it at some x, leaving two pieces. x and 10-x

send x to the square, each side is $\frac{x}{4}$ and 10-x to the circle with radius $r = \frac{10-x}{2\pi}$

$$A = A_{square} + A_{circle} = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{10-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(10-x)^2}{4\pi}$$

$$\implies A' = \frac{x}{8} - \frac{10-x}{2\pi} \text{ and } A' = 0 \iff x = \frac{40}{4+\pi}$$

$$\text{so } A(0) \approx 7.96\ ,\ A(10) \approx 6.25 \text{ and } A\left(\frac{40}{4+\pi}\right) \approx 3.5$$

3. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

inscribe a rectangle with y being one half the height, x being one half the base and let r

go from the center of the circle to the top right corner of the rectangle

$$A_{rectangle} = 2x2y = 4xy \text{ and since } r^2 = x^2 + y^2 \implies y = \sqrt{r^2 - x^2}$$

$$\implies A = 4x\sqrt{r^2 - x^2} \text{ and } A'(x) = 4\frac{r^2 - 2x^2}{\sqrt{r^2 - x^2}} \text{ and } A' = 0 \iff x = \frac{1}{\sqrt{2}}r$$

$$\implies \text{rectangle is a square}$$

1. If $x^3 - x^2 - 1 = 0$ and $x_1 = 1$, use Newton's method to find x_3.

$$f(x) = x^3 - x^2 - 1$$

$$f'(x) = 3x^2 - 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n^2 - 1}{3x_n^2 - 2x_n}$$

$$x_1 = 1 \ , \ x_2 = 2 \ , \ x_3 = 1.625$$

2. Use Newton's method to approximate $\sqrt[3]{30}$ correct to eight decimal places.

$$\text{If } \ x = \sqrt[3]{30} \ \text{ then } \ x^3 = 30 \implies x^3 - 30 = 0$$

$$f(x) = x^3 - 30$$

$$f'(x) = 3x^2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 30}{3x_n^2}$$

since 27 is close to 30 and $\sqrt[3]{27} = 3$ let $x_1 = 3$

$x_1 = 3 \quad x_2 \approx 3.11111111 \quad x_3 \approx 3.10723734 \quad x_4 \approx 3.10723251$

$x_5 \approx 3.10723251$ so $\sqrt[3]{30} \approx 3.10723251$ to 8 decimal places

MATH 150 **Section 6 Worksheet Key** **Exam 5**

1. Find f if $f''(x) = 2 + x^3 + x^6$.

$$f'(x) = 2x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + C$$

$$f(x) = x^2 + \frac{1}{20}x^5 + \frac{1}{56}x^8 + Cx + D$$

2. Find f if $f''(x) = 2 - 12x$, $f(0) = 9$ and $f(2) = 15$.

$$f'(x) = 2x - 6x^2 + C$$

$$f(x) = x^2 - 2x^3 + Cx + D$$

$$f(0) = D \quad \Longrightarrow \quad D = 9 \text{ and } f(2) = 2C - 3 \quad \Longrightarrow \quad C = 9$$

$$\text{thus } f(x) = x^2 - 2x^3 + 9x + 9$$

3. A particle is moving with velocity $v(t) = sin\ t - cos\ t$. Find its position if $s(0) = 0$.

$$s'(t) = sin\ t - cos\ t$$

$$s(t) = -cos\ t - sin\ t + C$$

$$s(0) = -1 + C \quad \Longrightarrow \quad C = 1 \text{ thus } s(t) = -cos\ t - sin\ t + 1$$