MATH 151 Worksheet Keys Exam 6

Worksheet for Section 1

1. Find the radius of convergence and interval of convergence of the series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+2} \frac{n+1}{x^n} \right| = \lim_{n\to\infty} \frac{|\, x\,|}{1+\frac{1}{n+1}} = |\, x \,|$$

by the Ratio Test the series converges when $|\, x \,| < 1 \implies R = 1$

when x=-1, divergent harmonic and when x=1, convergent alternating harmonic

$$I = (-1, 1]$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{5^{n+1}(n+1)^5} \frac{5^n n^5}{x^n} \right| = \lim_{n\to\infty} \frac{|\, x\,|}{5} \left(\frac{n}{n+1} \right)^5 = \frac{|\, x\,|}{5}$$

by the Ratio Test the series converges when $\frac{|\, x\,|}{5} < 1 \implies |\, x \,| < 5 \implies R = 5$

$$x = -5 \implies \sum_{n=1}^{\infty} \frac{(-1)^n}{n^5} \text{ converges by AST}$$

$$x = 5 \implies \sum_{n=1}^{\infty} \frac{1}{n^5} \text{ convergent p-series}$$

$$I = [-5, 5]$$

Worksheet for Section 2

1. Find a power series representation for the function and determine the interval of convergence.

   (a)
$$f(x) = \frac{3}{1-x^4}$$

$$3\left(\frac{1}{1-x^4}\right) \;=\; 3\sum_{n=0}^{\infty}(x^4)^n \;=\; \sum 3x^{4n}$$

$$\Longrightarrow \quad |\, x^4 \,| \;<1 \quad \Longrightarrow \quad |\, x \,| \;<1 \quad \Longrightarrow \quad R=1 \text{ and } I=(-1,1)$$

   (b)
$$f(x) = \frac{x}{4x+1}$$

$$x\left(\frac{1}{1-(-4x)}\right) \;=\; x\sum_{n=0}^{\infty}(-4x)^n \;=\; \sum (-1)^n 4^n x^{n+1}$$

$$\Longrightarrow \quad |\, -4x \,| \;<1 \quad \Longrightarrow \quad |\, x \,| \;< \frac{1}{4} \quad \Longrightarrow \quad I=\left(-\frac{1}{4},\frac{1}{4}\right)$$

2. Find a power series representation for $f(x) = ln(3+x)$

$$ln(3+x) \;=\; \int \frac{1}{3+x}\,dx \;=\; \frac{1}{3}\int \frac{1}{1+\frac{x}{3}}\,dx \;=\; \frac{1}{3}\int \frac{1}{1--\left(\frac{x}{3}\right)}\,dx$$

$$= \frac{1}{3}\int \sum \left(\frac{-x}{3}\right)^n dx \;=\; C+\frac{1}{3}\sum \frac{(-1)^n x^{n+1}}{(n+1)3^n}$$

$$f(0)=3 \quad \Longrightarrow \quad ln\ 3+\sum \frac{(-1)^{n-1}x^n}{n3^n}$$

Worksheet for Section 3

1. Find the Taylor series for $f(x) = ln\ x$ at $a = 2$.

$f(a) = ln\ 2$
$f'(x) = \frac{1}{x} \implies f'(2) = \frac{1}{2}$
$f''(x) = -\frac{1}{x^2} \implies f''(2) = -\frac{1}{4}$
$f'''(x) = \frac{2}{x^3} \implies f'''(2) = -\frac{2}{8}$

$$\text{so} \quad f(x) = ln\ 2 + \frac{1}{2}(x-2) - \frac{1}{4}\frac{(x-2)^2}{2!} + \frac{2}{8}\frac{(x-2)^3}{3!} + \ldots$$

$$\text{or} \quad f(x) = ln\ 2 + \sum \frac{(-1)^{n-1}(x-2)^n}{n \cdot 2^n}$$

2. Evaluate $\int \frac{sin\ x}{x}\, dx$ as an infinite series.

$$\frac{sin\ x}{x} = \frac{1}{x} \cdot \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\implies \int \frac{sin\ x}{x}\, dx = \int \sum \frac{(-1)^n x^{2n}}{(2n+1)!} = C + \sum \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+1)!}$$