MATH 254 Exam 3 Review Key Fall

1. If $p(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n$, define $p(A)$ to be the matrix formed by replacing each power of t in $p(t)$ by the corresponding power of A, with $A^0 = I$. That is,

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \ldots + c_n A^n$$

Show that if λ is an eigenvalue of A, then one eigenvalue of $p(A)$ is $p(\lambda)$.

Suppose that $Ax = \lambda x$ with $x \neq 0$. Then

$$\begin{aligned} p(A)x &= (c_0 I + c_1 A + c_2 A^2 + \ldots + c_n A^n)x \\ &= c_0 x + c_1 Ax + c_2 A^2 x + \ldots + c_n A^n x \\ &= c_0 x + c_1 \lambda x + c_2 \lambda^2 x + \ldots + c_n \lambda^n x = p(\lambda)x \end{aligned}$$

So $p(\lambda)$ is an eigenvalue of the matrix $p(A)$

2. Suppose $A = PDP^{-1}$, where P is 2×2 and $D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

   (a) Let $B = 5I - 3A + A^2$. Show that B is diagonalizable by finding a suitable factorization of B.

   If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$, and

$$B = 5I - 3A + A^2 = 5PIP^{-1} - 3PDP^{-1} + PD^2P^{-1} = P(5I - 3D + D^2)P^{-1}$$

   Since D is diagonal, so is $5I - 3D + D^2$. Thus, B is similar to a diagonal matrix.

   (b) Given $p(t)$ and $p(A)$ from the previous exercise, show that $p(A)$ is diagonalizable.

$$\begin{aligned} p(A) &= c_0 I + c_1 PDP^{-1} + c_2 PD^2P^{-1} + \ldots + c_n PD^nP^{-1} \\ &= P(c_0 I + c_1 D + c_2 D^2 + \ldots + c_n D^n)P^{-1} \\ &= Pp(D)P^{-1} \end{aligned}$$

This shows that $p(A)$ is diagonalizable, because $p(D)$ is a linear combination of diagonal matrices and therefore is diagonal. In fact,

$$p(D) = \begin{bmatrix} p(2) & 0 \\ 0 & p(7) \end{bmatrix}$$

3. Suppose A is diagonalizable and $p(t)$ is the characteristic polynomial of A. Define $p(A)$ as in exercise 1, and show that $p(A)$ is the zero matrix. This fact, which is also true for *any* square matrix, is called the *Cayley-Hamilton Theorem*.

If $A = PDP^{-1}$, then $p(A) = Pp(D)P^{-1}$ as was shown previously. If the (j, j) entry in D is λ, then the (j, j) entry in D^k is λ^k and the (j, j) entry in $p(D)$ is $p(\lambda)$. If p is the characteristic polynomial of A, then $p(A) = 0$ for each diagonal entry of D, because these entries in D are the eigenvalues of A. Thus, $p(D)$ is the zero matrix and $p(A) = P(0)P^{-1} = 0$

4. (a) Let A be a diagonalizable $n \times n$ matrix. Show that if the multiplicity of an eigenvalue λ is n, then $A = \lambda I$.

   If λ is an eigenvalue of an $n \times n$ diagonalizable matrix A, then $A = PDP^{-1}$ for an invertible matrix P and an $n \times n$ diagonal matrix D whose diagonal entries are the eigenvalues of A. If the multiplicity of λ is n, then λ must appear in every entry of D. That is, $D = \lambda I$. In this case, $A = P(\lambda I)P^{-1} = \lambda PIP^{-1} = \lambda PP^{-1} = \lambda I$

   (b) Use part (a) to show that the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable.

Since the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is triangular, its eigenvalues are on the diagonal. Thus 3 is an eigenvalue with multiplicity 2. If the 2×2 matrix A were diagonalizable, then A would be $3I$ by part (a). This is not the case, so A is not diagonalizable.

5. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. The trace of A, denoted $tr\ A$, is the sum of the diagonal entries in A. Show that the characteristic polynomial of A is $\lambda^2 - (tr\ A)\lambda + det\ A$. Then show that the eigenvalues of a 2×2 matrix A are both real if and only if

$$det\ A \leq \left(\frac{tr\ A}{2}\right)^2$$

$det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 - (tr\ A)\lambda + det\ A$
Using the quadratic formula:

$$\lambda = \frac{tr\ A \pm \sqrt{(tr\ A)^2 - 4\ det\ A}}{2}$$

The eigenvalues are both real if and only if the discriminant is nonnegative, that is, $(tr\ A)^2 - 4\ det\ A \geq 0$. Simplifying we get

$$(tr\ A)^2 \geq 4\ det\ A \implies det\ A \leq \left(\frac{tr\ A}{2}\right)^2$$