MATH 254 Exam 3 Review Fall

1. If $p(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n$, define $p(A)$ to be the matrix formed by replacing each power of t in $p(t)$ by the corresponding power of A, with $A^0 = I$. That is,

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \ldots + c_n A^n$$

Show that if λ is an eigenvalue of A, then one eigenvalue of $p(A)$ is $p(\lambda)$.

2. Suppose $A = PDP^{-1}$, where P is 2×2 and $D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

   (a) Let $B = 5I - 3A + A^2$. Show that B is diagonalizable by finding a suitable factorization of B.

   (b) Given $p(t)$ and $p(A)$ from the previous exercise, show that $p(A)$ is diagonalizable.

3. Suppose A is diagonalizable and $p(t)$ is the characteristic polynomial of A. Define $p(A)$ as in exercise 1, and show that $p(A)$ is the zero matrix. This fact, which is also true for *any* square matrix, is called the *Cayley-Hamilton Theorem.*

4. (a) Let A be a diagonalizable $n \times n$ matrix. Show that if the multiplicity of an eigenvalue λ is n, then $A = \lambda I$.

   (b) Use part (a) to show that the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable.

5. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. The trace of A, denoted $tr\ A$, is the sum of the diagonal entries in A. Show that the characteristic polynomial of A is $\lambda^2 - (tr\ A)\lambda + det\ A$. Then show that the eigenvalues of a 2×2 matrix A are both real if and only if

$$det\ A \leq \left(\frac{tr\ A}{2}\right)^2$$