MATH 245 **Exam 3 Notes** **Spring**

1 Recurrence Relations

Definition 1

A *recurrence relation* defines a sequence by giving the nth value in terms of its predecessors. You also need a starting point called an *initial condition.*

Famous Example 1

THE FIBONACCI SEQUENCE

$$f_n = f_{n-1} + f_{n-2} \qquad n \geq 3$$
$$f_1 = 1 \qquad f_2 = 2$$

So the sequence is 1,2,3,5,8,...

Famous Example 2

THE TOWERS OF HANOI

http://www.mathsisfun.com/games/towerofhanoi.html

It turns out that the minimum number of moves to solve the puzzle with n disks is:

$$c_n = 2c_{n-1} + 1 \qquad n > 1\ , \quad c_1 = 1$$

The idea here is to solve these recurrence relations by finding the pattern. In the Tower of Hanoi example, if we would like to know the minimum number of moves with 6 disks, do we need to go through the sequence five times?

ex 1

Let $a_n = a_{n-1} + 3$ with $a_1 = 2$. Find the closed form solution.

So

$$\begin{aligned} a_1 &= 2 \\ a_2 &= a_1 + 3 = 2 + 3 \\ a_3 &= a_2 + 3 = 2 + 3 + 3 = 2 + 2(3) \\ a_4 &= a_3 + 3 = (2 + 3 + 3) + 3 = 2 + 3(3) \\ &\vdots \\ a_n &= 2 + (n-1)3 \end{aligned}$$

This is a closed form solution since we can now go straight to $n = 10$ without having to calculate $n = 1, 2, 3, \ldots$

$$a_{10} = 2 + (10 - 1)3 = 29$$

ex 2

Why don't you now try to solve the Towers of Hanoi at your desks. Recall that

$$c_n = 2c_{n-1} + 1 \qquad c_1 = 1$$

Definition 2

A linear homogeneous recurrence relation of order k with constant coefficients is of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

with $c_k \neq o$ and k initial conditions.

Theorem 1

Let $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$ with $a_0 = c_0$ and $a_1 = c_1$.

If r is a root of $t^2 - c_1 t - c_2 = 0 \quad r_1 \neq r_2$ then the solution is of the form

$a_n = b \cdot r_1^n + d \cdot r_2^n \qquad$ for $\quad n = 0, 1, \ldots \qquad$ where b and d are constants

If $r_1 = r_2$, then

$a_n = b \cdot r_1^n + n \cdot d \cdot r_2^n \qquad$ for $\quad n = 0, 1, \ldots \qquad$ where b and d are constants

ex 3

Solve the following:

$$a_n = 5 \cdot a_{n-1} - 6 \cdot a_{n-2} \quad \text{with } a_0 = 7 \text{ and } a_1 = 16$$

Let $a_n = t^n$. Then

$$t^n = 5t^{n-1} - 6t^{n-2} \iff t^n - 5t^{n-1} + 6t^{n-2} = 0 \iff t^2 - 5t + 6 = 0$$

So $t = 2$ or $t = 3$. Thus, the general solution is:

$$a_n = b \cdot 2^n + d \cdot 3^n$$

Since $a_0 = 7$ we have $7 = b \cdot 2^0 + d \cdot 3^0 \implies b + d = 7$ and

Since $a_1 = 16$ we have $16 = b \cdot 2^1 + d \cdot 3^1 \implies 2b + 3d = 16$

thus, $b = 5$ and $d = 2$ and the general solution is

$$5 \cdot 2^n + 2 \cdot 3^n \quad \text{for} \quad n = 0, 1, \ldots$$

Worksheet for Section 1

1. Solve the Fibonacci Sequence. That is, solve

$$f_n - f_{n-1} - f_{n-2} = 0$$

for $n \geq 3$ where $f_1 = 1$ and $f_2 = 2$.

2. Solve

$$d_n = 4d_{n-1} - 4d_{n-2}$$

where $d_0 = d_1 = 1$.