1 Modeling with Differential Equations

Recall that the derivative represents a rate of change. In order to form mathematical models we usually observe the rate of change and want to use this to predict future change.

That is, we are observing the derivative.

This section will deal with *differential equations*. Equations that contain an unknown function and some of its derivatives.

Population growth is one such example. Let's assume that, under ideal conditions, a population grows at a rate proportional to its size. In other words, the larger the population, the faster the growth. Let t represent time (the independent variable) and P represent the number in the population (the dependent variable).

So the rate of growth can be expressed as

$$\frac{dP}{dt}$$

By our assumption

$$\frac{dP}{dt} = kP$$

where k is a constant of proportionality.

This equation is an example of a differential equation. It has an unknown function, P and its derivative,

$$\frac{dP}{dt}$$

If $k > 0 \implies P'(t) > 0 \implies$ *population is always increasing*

Can you think of a solution to our differential equation?

In other words, is there a function whose derivative is a multiple of itself?

exponential functions have this property

Let

$$P(t) = Ce^{kt}$$

then

$$P'(t) \; = \; C\; ke^{kt} \; = \; k\; Ce^{kt} \; = \; k\; P(t)$$

So any exponential function of the form $P(t) = Ce^{kt}$ is a solution.

If $y = Ce^{kt} \implies y' = ky$

$$or \qquad \frac{y'}{y} = k$$

integrating with respect to t yields

$$\int \frac{y'}{y}\, dt \; = \; \int k\; dt$$

but $dy = y'\; dt$ so

$$\int \frac{1}{y}\, dy \; = \; \int k\; dt \quad \Longleftrightarrow \quad ln \mid y \mid \; = \; kt + C_1$$

$$\implies \quad y = e^{kt}e^{C_1} \quad \implies \quad y = Ce^{kt}$$

As C varies through the reals we get a family of solutions. Does this really happen this way? No.

Realistically populations begin to increase exponentially but begin to level off when they reach what is called their *carrying capacity*, K.

We need two facts:

1.
$$\frac{dP}{dt} \approx kP \;\; if \; P \; is \; small$$

2.
$$\frac{dP}{dt} < 0 \;\; if \;\; P > K$$

Taking both into account yields what is called the ***Logistical Differential Equation***

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

$$P \; small \implies \frac{P}{K} \; is \; close \; to \; 0 \implies \frac{dP}{dt} \approx kP$$

$$P > K \implies \left(1 - \frac{P}{K}\right) < 0 \implies \frac{dP}{dt} < 0$$

Note that $P(t) = 0$ and $P(t) = K$ are solutions. Why do you think this is?

If a population is ever zero or its carrying capacity then it stays that way.

$P(t) = 0$ and $P(t) = K$ are called ***equilibrium solutions***

Another example involves springs. Hooke's Law says that if a spring is stretched x units from its natural length then

$$restoring\ force \;=\; -kx$$

where k is a positive spring constant.

$$force \;=\; ma \quad \Longrightarrow \quad m\,\frac{d^2x}{dt^2} \;=\; -kx \quad OR \quad \frac{d^2x}{dt^2} \;=\; -\frac{k}{m}\,x$$

This is an example of a second order differential equation. Why do you think second order? Can you guess a general solution?

What type of function has a second derivative that is proportional to itself but of the opposite sign? Yes, that was on purpose.

A Differential Equation

is an equation containing an unknown function and one or more of its derivatives.

The Order

of a differential equation is the order of the highest derivative that occurs in the equation.

A function is a *solution* of a differential equation if the equation is satisfied when $f(x)$ and its derivatives are substituted.

ex 1 Is the function $y = 4e^{-x}$ a solution to the differential equation $y'' - y = 0$?

All you need to do is verify, that is, when you plug in y and y'', does it satisfy the equation? The answer is yes.

Usually we will be solving for a particular solution, when $y(t_0) = y_0$ for example. This is called an *initial condition* and the problem is then referred to as an *initial-value problem* or IVP.

ex 2 For the differential equation $xy' - 3y = 0$, verify that $y = Cx^3$ is a solution and find the particular solution determined by the initial condition $y(-3) = 2$.

$$y = Cx^3 \quad \Longrightarrow \quad 2 = C(-3)^3 \quad \Longrightarrow \quad C = -\frac{2}{27}$$

Worksheet for Section 1

1. Show that every member of the family of functions $y = Ce^{x^2/2}$ is a solution of the differential equation $y' = xy$.

2. Find a solution of the differential equation $y' = xy$ that satisfies the initial condition $y(0) = 5$.

Homework for Section 1

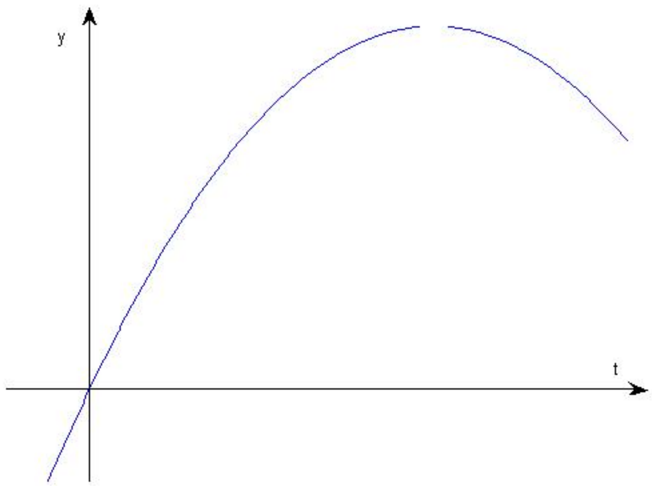
1. Which of the following functions are solutions of the differential equation $y'' + y = sin\ x$?
   (a) $y = sin\ x$
   (b) $y = cos\ x$
   (c) $y = \frac{1}{2}x\ sin\ x$
   (d) $y = -\frac{1}{2}x\ cos\ x$
2. Given the differential equation $y' = -y^2$:
   (a) What can you say about a solution by just looking at the differential equation?
   (b) Verify that all members of the family $y = 1/(x + C)$ are solutions.
   (c) Can you think of a solution that is not a member of the family in part (b)?
   (d) Find a solution to $y' = -y^2 \qquad y(0) = 0.5$
3. A population is modeled by the differential equation

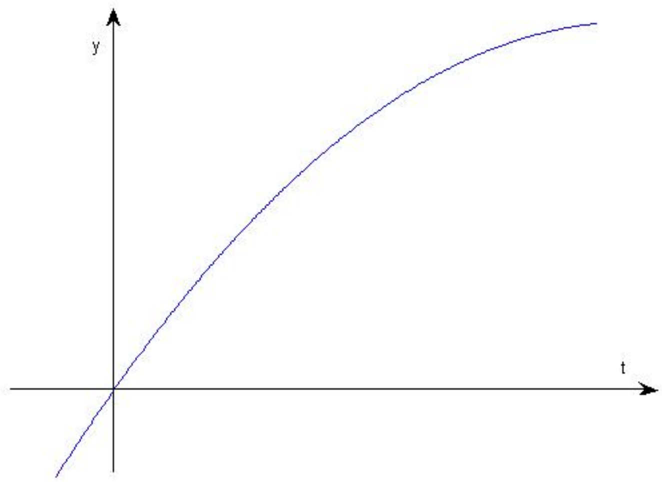
$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

   (a) For what values of P is the poulation increasing?
   (b) For what values of P is the poulation decreasing?
   (c) What are the equilibrium solutions?

4. Explain why the functions with the given graphs can NOT be solutions to the following:

$$\frac{dy}{dt} = e^t(y-1)^2$$





2 Approximations

Unfortunately it is impossible to solve most differential equations explicitly thus we must....you guessed it **approximate**

We can do this graphically $\Longrightarrow$ *DIRECTION FIELDS*

or

We can do this numerically $\Longrightarrow$ *EULER'S METHOD*

2.1 Direction Fields

or sometimes these are called slope fields.

ex 3 Sketch a solution to the initial value problem $y' = x + y$ with $y(0) = 1$

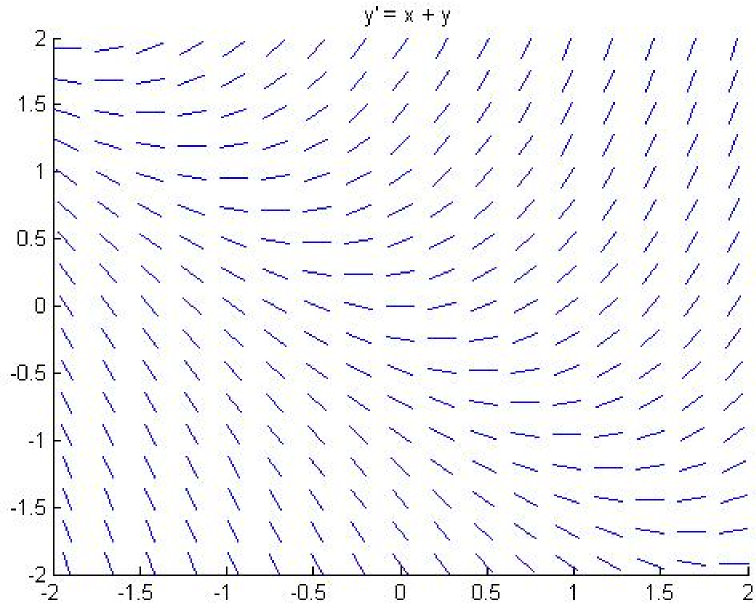
$y' = x + y \quad says \quad \Longrightarrow \quad slope = x\ coordinate \ + \ y\ coordinate$

so,

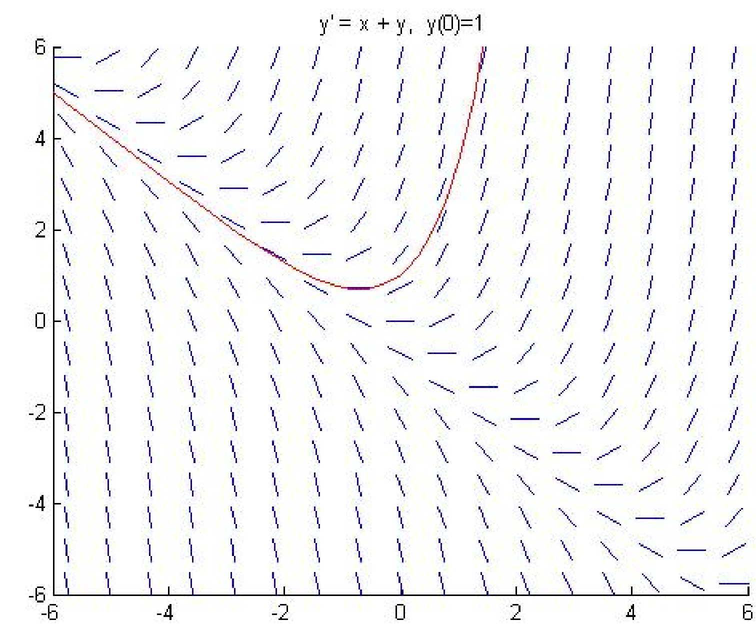
x	-1	0	1	2	-1	0	1	2
y	0	0	0	0	1	1	1	1
y'=x+y	-1	0	1	2	0	1	2	3

. . .

Plotting tiny lines with the appropriate slopes at these points yields the following direction field



Since we know that our solution goes through $(0, 1)$ we get the following curve



2.2 Euler's Method

This method is the numerical equivalent to direction fields. Let's use the same example as before.

So,

$y' = x + y \quad and \quad y(0) = 1$

Since $y'(0) = 0 + 1$ we know that the solution curve has a slope of 1 at $(0, 1)$

Let's use a *linear approximation*, or tangent line approximation, to estimate the solution.

$$L(x) = x + 1$$

This is a linear function with slope 1 that goes through the point $(0, 1)$

Euler's Method says, proceed a short distance along the tangent line and then make a mid-course correction according to the direction field. This horizontal distance is known as the *step size*.

If we proceed from $x = 0$ to $x = \frac{1}{2}$ then $L\left(\frac{1}{2}\right) = \frac{3}{2}$

so use $\left(\frac{1}{2}, \frac{3}{2}\right)$ as the starting point for a NEW line segment

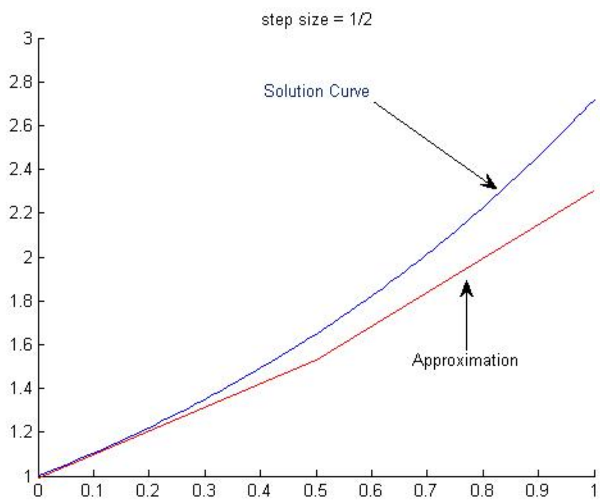
The differential equation says

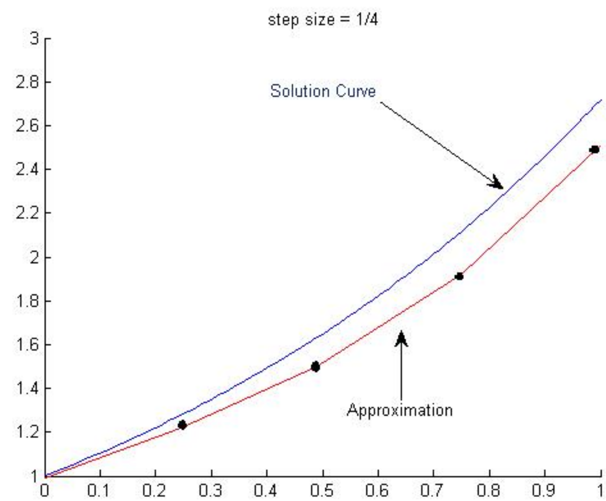
$$y'\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{3}{2} = 2$$

So to approximate the solution curve for $x > \frac{1}{2}$ use the line

$$y - \frac{3}{2} = 2\left(x - \frac{1}{2}\right) \quad \Longrightarrow \quad y = 2x + \frac{1}{2}$$

- Here are two different approximations, one with $h = 1/2$ and the other with $h = 1/4$





In general, when h represents the step size and $F(x, y)$ represents y', we get:

$$y_1 = y_0 + hF(x_0, y_0)$$
$$y_2 = y_1 + hF(x_1, y_1)$$
$$\vdots$$
$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

ex 4 Use Euler's method with step size .1 to approximate $y(.3)$ if $y' = x + y$ and $y(0) = 1$

Since $F(x, y) = x + y$ we obtain

$$y_1 = y_0 + hF(x_0, y_0) \;=\; 1 + .1(0 + 1) \;=\; 1.1$$

$$y_2 = 1.1 + .1(.1 + 1.1) \;=\; 1.22$$

$$y_3 = 1.22 + .1(.2 + 1.22) \;=\; 1.362$$

$$\Longrightarrow \quad y(.3) \;\approx\; 1.362$$

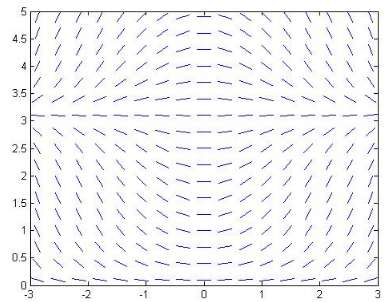
Worksheet for Section 2

1. Sketch a direction field for the differential equation $y' = 1 + y$ and sketch a solution curve through the point $(0, 0)$.

2. Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial value problem $y' = 1 - xy$, $y(0) = 0$.

Homework for Section 2

1. A direction field for the differential equation $y' = x \ sin \ y$ is shown.



   (a) Sketch the graphs of the solutions that satisfy the given initial conditions.

      i. $y(0) = 1$

      ii. $y(0) = 2$

      iii. $y(0) = \pi$

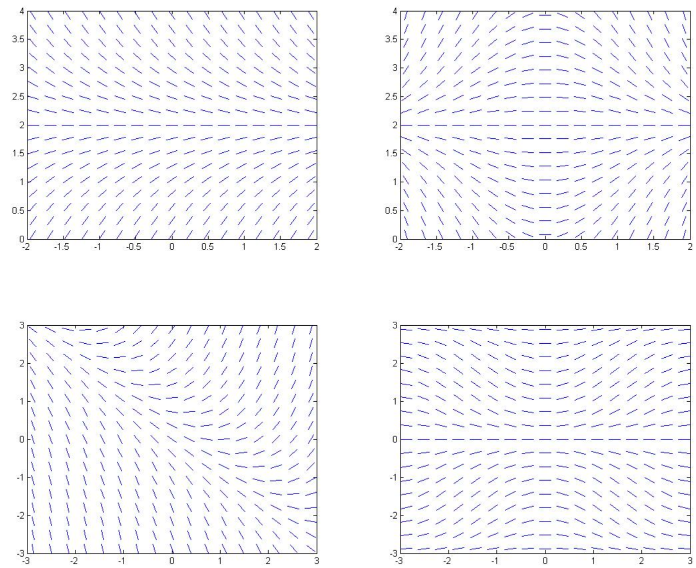
      iv. $y(0) = 4$

      v. $y(0) = 5$

   (b) Find all of the equilibrium solutions.

2. Use Euler's method with step size 0.1 to estimate $y(0.5)$ where y is the solution to $y' = y + xy \ \ , \ \ y(0) = 1$

3. Given the following direction fields, match them with their differential equations.



(a) $y' = 2 - y$

(b) $y' = x(2 - y)$

(c) $y' = x + y - 1$

(d) $y' = sin\ x\ sin\ y$

3 Separable Equations

Can we solve the following Initial Value Problem (IVP)

$$y' = \frac{x}{\sqrt{x^2+9}} \qquad y(4) = 2$$

Actually integrating both sides works in this case, that is

$$\int y'\,dx \;=\; \int \frac{x}{\sqrt{x^2+9}}\,dx \;\Longrightarrow\; y = \sqrt{x^2+9}+C \;\Longrightarrow\; y = \sqrt{x^2+9}-3$$

If you recall, when given an acceleration how did we find the position? We integrated twice.

These two examples involve *Direct Integration*

A similar technique is *Separation of Variables*

A *first order* differential equation is *separable* if it can be written as a product of a function x and a function y, that is

$$F(x,y) \;=\; \frac{dy}{dx} \;\; \textit{can be written as} \;\; \frac{dy}{dx} \;=\; g(x)h(y) \;=\; \frac{g(x)}{f(y)}$$

$$\textit{where} \;\; h(y) \;=\; \frac{1}{f(y)}$$

In this case the variables x and y can be separated or isolated on opposite sides

So, written **INFORMALLY**

$$f(y)dy \;=\; g(x)dx$$

- Which we understand to be concise notation for the differential equation

$$f(y)\frac{dy}{dx} \;=\; g(x)$$

Then we integrate both sides of $f(y)dy \;=\; g(x)dx$

Justification for this is derived from the chain rule.

$$To\, see\, that \;\; \int f(y)\,dy \;=\; \int g(x)\,dx \;\; is\, the\, same\, as \;\; f(y)\frac{dy}{dx} \;=\; g(x) \;\; \ldots$$

$$\frac{d}{dx}\left(\int f(y)\,dy\right) \;=\; \frac{d}{dx}\left(\int g(x)\,dx\right)$$

$$\frac{d}{dy}\left(\int f(y)\,dy\right)\frac{dy}{dx} \;=\; g(x) \qquad\qquad (by\ the\ chain\ rule)$$

$$So \;\; f(y)\frac{dy}{dx} \;=\; g(x)$$

ex 5 Solve the IVP $\frac{dy}{dx} = -6xy$ with $y(0) = 7$

Informally:

$$\int \frac{dy}{y} \;=\; \int -6x\;dx \quad\Longrightarrow\quad ln\mid y\mid = -3x^2 + C \quad\Longrightarrow$$

$$y = Ae^{-3x^2} \quad\Longrightarrow\quad y = 7e^{-3x^2}$$

ex 6 Solve the following differential equation

$$x^2y' \;=\; \frac{x^2+1}{3y^2+1}$$

$$\int (3y^2+1)\ dy \ = \ \int \left(1+\frac{1}{x^2}\right)\ dx \quad \Longrightarrow \quad y^3+y=x-\frac{1}{x}+C$$

This is called an *implicit solution.* Take a guess as to why that is.

ex 7 Solve the IVP $\frac{dy}{dx}=xy+x-2y-2$ with $y(0)=2$

$$\frac{dy}{dx} \ = \ (x-2)(y+1) \quad \Longrightarrow \quad \int \frac{dy}{y+1} \ = \ \int (x-2)\ dx \quad \Longrightarrow$$

$$ln \mid y+1 \mid = \frac{1}{2}x^2-2x+C$$

$$\Longrightarrow \quad y=Ae^{x^2/2-2x}-1 \quad \textit{and since} \quad y(0)=2 \quad \Longrightarrow \quad y=3e^{x^2/2-2x}-1$$

Worksheet for Section 3

1. Solve the differential equation: $(x^2 + 1)y' = xy$
2. Solve the initial value problem: $x \; cos \; x = (2y + e^{3y})y' \; , \; y(0) = 0$

Homework for Section 3

1. Solve the following differential equations.

   (a)
   $$\frac{dy}{dx} = \frac{y}{x}$$

   (b)
   $$(x^2 + 1)y' = xy$$

   (c)
   $$(1 + tan\ y)y' = x^2 + 1$$

   (d)
   $$\frac{dy}{dt} = \frac{te^t}{y\sqrt{1 + y^2}}$$

   (e)
   $$\frac{dy}{dx} = \frac{x}{y} \quad , \quad y(0) = -3$$

   (f)
   $$x\ cos\ x = (2y + e^{3y})y' \quad , \quad y(0) = 0$$

2. Given the following, $\frac{dP}{dt} = k(M - P)$, where $P(t)$ is the performance after training time t, M is the maximum level of performance and k is a positive constant. Solve this differential equation for P and find the limit of that expression.

4 Population Growth

Recall:

$$\frac{dP}{dt} \;=\; kP\left(1-\frac{P}{K}\right)$$

So separating we get

$$\int \frac{dP}{P\left(1-\frac{P}{K}\right)} \;=\; \int k\ dt$$

Note that

$$\frac{1}{P\left(1-\frac{P}{K}\right)} \;=\; \frac{K}{P(K-P)}$$

And by partial fractions

$$\Longrightarrow \quad \frac{K}{P(K-P)} \;=\; \frac{1}{P} \;+\; \frac{1}{K-P}$$

So

$$\int \frac{1}{P}\ dP \;+\; \int \frac{1}{K-P}\ dP \;=\; \int k\ dt$$

$$\Longrightarrow \quad ln \mid P \mid - ln \mid K-P \mid = kt+C \quad \Longrightarrow \quad ln \left| \frac{K-P}{P} \right| = -kt+C$$

$$\Longrightarrow \quad \frac{K-P}{P} = Ae^{-kt} \quad \Longrightarrow \quad P \;=\; \frac{K}{1+Ae^{-kt}}$$

So the solution to the logistical equation is

$$P(t) \;=\; \frac{K}{1 + Ae^{-kt}} \quad where \quad A \;=\; \frac{K - P_0}{P_0}$$

Since

$$when\ t = 0 \;\Longrightarrow\; P_0 = P\ \ and\ \ Ae^{-kt} = \frac{K - P}{P} \;\Longrightarrow\; A = \frac{K - P_0}{P_0}$$

ex 8 A lake is stocked with 400 fish and has a carrying capacity of 10,000. The number of fish tripled in the first year. Find an expression for the size of the population after t years.

So

$$P(0) = P_0 = 400 \quad and \quad P(1) = 1200 \quad\Longrightarrow\quad A = \frac{K - P_0}{P_0} = 24$$

$$P \;=\; \frac{10,000}{1 + 24e^{-kt}} \quad and\ since \quad P(1) = 1200 \quad\Longrightarrow\quad k = ln\,\frac{36}{11}$$

Worksheet for Section 4

Show that if P satisfies the logistic equation, then

1. $\frac{d^2P}{dt^2} = k^2 P \left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{K}\right)$
2. Show that a population grows fastest when it reaches half its carrying capacity.

Homework for Section 4

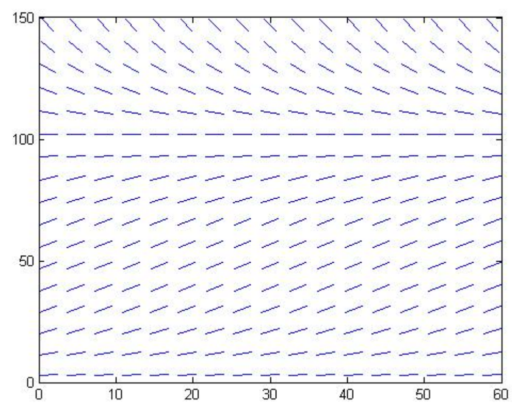
1. Suppose a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 0.0005P^2$$

where t is measured in weeks.

(a) What is the carrying capacity?

(b) Given the direction field, sketch solutions for initial populations of 20,40,60,80,120 and 140.



(c) What are the equilibrium solutions?

5 Linear Equations

Recall:

$$\textit{to solve} \quad \frac{dy}{dx} = 2xy$$

We can multiply both sides by $\frac{1}{y}$ and get:

$$\frac{dy}{dx} \bullet \frac{1}{y} = 2x$$

Why? Then each side is recognizable as a *derivative* with respect to x

$$D_x[ln\ y] = D_x[x^2]$$

All that remains is simple integration

- *The function* $\frac{1}{y}$ *is called an* **integrating factor** •

Definition

An *integrating factor* for an ODE is a function such that multiplication by each side yields an equation where each side is recognizable as a derivative.

So, with the aid of an appropriate integrating factor, there is a standard technique for solving a *linear first order equation.* That is,

$$\frac{dy}{dx} + P(x) \bullet y = Q(x)$$

on an interval where $P(x)$ and $Q(x)$ are continuous

You multiply BOTH sides by the integrating factor $\rho(x)$ where:

$$\rho(x) \; = \; e^{\int P(x)\, dx}$$

Why? That forces the left hand side of the resulting equation to be the **derivative** of the product $\rho(x) \bullet y(x)$

ex 9 Solve $x^2y' + xy = 1$ for $x > 0$

So,

$$y' + \frac{1}{x} \bullet y = \frac{1}{x^2} \quad and \quad \rho(x) = e^{\int \frac{1}{x}\, dx} \; = \; e^{\,ln\ x} \; = \; x$$

Note that we MUST put the equation in STANDARD FORM first if we are to find $\rho(x)$

Thus

$$xy' + y = \frac{1}{x} \quad or \quad (xy)' = \frac{1}{x} \quad since \quad (xy)' = xy' + y \quad from\, the\, product\, rule$$

$$\Longrightarrow \quad \int (xy)'\, dx \; = \; \int \frac{1}{x}\, dx \quad \Longrightarrow \quad xy = ln \mid x \mid + C$$

$$\Longrightarrow \quad y \; = \; \frac{ln \mid x \mid \; + C}{x}$$

ex 10 Solve the IVP $\frac{dy}{dx} - 3y = e^{2x}$ with $y(0) = 3$

This is already in standard form so,

$$\rho(x) = e^{\int -3\, dx} = e^{-3x} \implies e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-x}$$

Or

$$\begin{aligned} \frac{d}{dx}\left(e^{-3x}y\right) &= e^{-x} \\ \implies e^{-3x}y &= \int e^{-x}\, dx = -e^{-x} + C \end{aligned}$$

Thus

$$y = Ce^{3x} - e^{2x} \quad \textit{and since} \quad y(0) = 3 \implies C = 4 \implies y = 4e^{3x} - e^{2x}$$

Worksheet for Section 5

1. Solve the differential equation: $y' + 2y = 2e^x$
2. Solve the initial value problem: $\frac{dv}{dt} - 2tv = 3t^2\, e^{t^2}\ , \quad v(0) = 5$

Homework for Section 5

1. Determine whether the following differential equations are linear.

   (a) $y' + cos\ x = y$

   (b) $yy' + xy = x^2$

2. Solve the following:

   (a) $y' + 2y = 2e^x$

   (b) $xy' - 2y = x^2$

   (c) $xy' + y = \sqrt{x}$

   (d) $y' = x + y \quad , \quad y(0) = 2$

   (e) $\frac{dv}{dt} - 2tv = 3t^2 e^{t^2} \quad , \quad v(0) = 5$