MATH 150 **Exam 4 Notes** **Fall**

1 Derivatives of Log Functions

$$\frac{d}{dx}(log_a\ x) = \frac{1}{x\ ln\ a}$$

What do you think the chain rule would do here?

$$\frac{d}{dx}(log_a\ (f(x))) = \frac{1}{f(x)\ ln\ a} \cdot f'(x) = \frac{f'(x)}{f(x) ln\ a}$$

Let's show why this is indeed true

If $y = log_a\ x$ then $a^y = x$ by definition. We shall use ID on $a^y = x$

$$\frac{d}{dx}[a^y = x] \iff a^y \cdot ln\ a \cdot \frac{dy}{dx} = 1$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{a^y\ ln\ a} = \frac{1}{x\ ln\ a}\ \ since\ \ a^y = x$$

If $a = e$ we get

$$\frac{d}{dx}(ln\ x) = \frac{1}{x}$$

and

$$\frac{d}{dx}(ln\ f(x)) = \frac{f'(x)}{f(x)}$$

ex 1 Find y' if $y = ln(x^2 + 1)$

$$\frac{d}{dx}(ln\ f(x)) = \frac{f'(x)}{f(x)} = \frac{2x}{x^2+1}$$

ex 2 Find y' if $y = ln(sin\ x)$

$$\frac{d}{dx}(ln\ f(x)) = \frac{f'(x)}{f(x)} = \frac{cos\ x}{sin\ x} = cot\ x$$

ex 3 Find y' if

$$y = ln\ \left(\frac{x+1}{\sqrt{x-2}}\right)$$

$$\frac{d}{dx}(ln\ f(x)) = \frac{f'(x)}{f(x)} = \frac{\frac{d}{dx}\left(\frac{x+1}{\sqrt{x-2}}\right)}{\frac{x+1}{\sqrt{x-2}}} = ugly\ mess$$

Is there an easier way?

Use your log properties!

$$y = ln\ \left(\frac{x+1}{\sqrt{x-2}}\right) = ln(x+1) - ln((x-2)^{1/2}) = ln(x+1) - \frac{1}{2}ln(x-2)$$

$$\implies y' = \frac{1}{x+1} - \frac{1}{2(x-2)}$$

This is the perfect transition into LOGARITHMIC DIFFERENTIATION or LD

1. Take the natural log of both sides and simplify using the log rules
2. Differentiate **implicitly** with respect to x
3. Solve for y' or $\frac{dy}{dx}$, whichever you used

We can use LD for a much more elegant proof of the power rule ...

If $y = x^n$ then

$$ln\ y \ = \ n \cdot ln\ x \iff \frac{y'}{y} = \frac{n}{x} \iff y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

BE CAREFUL

If a and b are constants

1. $\frac{d}{dx}(a^b) \ = \ 0$
2. $\frac{d}{dx}[f(x)]^b \ = \ b[f(x)]^{b-1} \cdot f'(x)$
3. $\frac{d}{dx}(a^{g(x)}) \ = \ a^{g(x)} \cdot ln\ a \cdot g'(x)$
4. $\frac{d}{dx}[f(x)]^{g(x)} \implies$ use LD

ex 4 Differentiate

$$y = x^{\sqrt{x}}$$

Here we need to use LD, so

$$ln\, y \ = \ ln\, x^{\sqrt{x}} \iff ln\, y \ = \ \sqrt{x}\, ln\, x \iff \frac{y'}{y} \ = \ (\sqrt{x})\frac{1}{x} + (ln\, x)\frac{1}{2\sqrt{x}}$$

$$\implies y' \ = \ y\left[\frac{1}{\sqrt{x}} + \frac{ln\ x}{2\sqrt{x}}\right] \ = \ x^{\sqrt{x}}\left[\frac{1}{\sqrt{x}} + \frac{ln\ x}{2\sqrt{x}}\right]$$

ex 5 Differentiate

$$y = \sqrt{\frac{x^2+1}{x^2-1}}$$

Do we need to use LD here? No, but why should we?

$$\begin{aligned} ln\ y &= ln\ \sqrt{\frac{x^2+1}{x^2-1}} \iff \\ ln\ y &= \frac{1}{2}\ ln\ \frac{x^2+1}{x^2-1} \iff \\ ln\ y &= \frac{1}{2}\left[ln\ (x^2+1) - ln\ (x^2-1)\right] \end{aligned}$$

$$\Longrightarrow \frac{y'}{y} = \frac{1}{2}\left[\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right] \Longrightarrow y' = y\left[\frac{x}{x^2+1} - \frac{x}{x^2-1}\right]$$

$$\Longrightarrow y' = \sqrt{\frac{x^2+1}{x^2-1}}\left[\frac{x}{x^2+1} - \frac{x}{x^2-1}\right]$$

Use the log properties to your advantage if you can.

Since we are on the subject, here are two interesting ways to represent e as a limit

$$If\ \ f(x) = ln\ x \Longrightarrow f'(x) = \frac{1}{x}\ \ and\ \ f'(1) = 1$$

So, by definition

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{x\to 0} \frac{f(1+x)-f(1)}{x} =$$

$$\lim_{x\to 0} \frac{ln(1+x)-ln(1)}{x}$$

$$= \lim_{x\to 0} \frac{ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{x}\cdot ln(1+x) =$$

$$lim_{x\to 0}\ ln(1+x)^{\frac{1}{x}} \ = \ 1$$

Thus

$$\lim_{x\to 0}\ ln(1+x)^{\frac{1}{x}} \ = \ 1$$

Now, exponential functions are continuous so

$$e \ = \ e^1 \ = \ e^{\lim_{x\to 0}\ ln(1+x)^{\frac{1}{x}}} \ = \ \lim_{x\to 0}\ e^{ln(1+x)^{\frac{1}{x}}}$$

$$= \ \lim_{x\to 0}\ (1+x)^{\frac{1}{x}}$$

Thus

$$e \ = \ \lim_{x\to 0}\ (1+x)^{\frac{1}{x}}$$

$$or$$

$$e \ = \ \lim_{n\to\infty}\ \left(1+\frac{1}{n}\right)^n$$

Worksheet for Section 1

1. Find y' and y'' if
$$y = \frac{ln\ x}{x^2}$$

2. Use logarithmic differentiation to find y' if $y = (sin\ x)^x$

3. Differentiate:
$$f(x) = ln\left(\frac{x}{x-1}\right)$$

Homework for Section 1

1. Differentiate the following:

   (a) $f(x) = log_2\,(1 - 3x)$

   (b) $f(x) = \sqrt[5]{ln\ x}$

   (c) $f(x) = sin\ x\ ln(5x)$

   (d)
   $$f(x) = ln\left(\frac{(2x+1)^3}{(3x-1)^4}\right)$$

   (e) $f(x) = ln(x\sqrt{x^2 - 1})$

2. Find y' and y'':

   (a) $y = x^2 ln(2x)$

   (b)
   $$y = \frac{ln\ x}{x^2}$$

3. Use LD to find the derivative of the following:

   (a) $y = (2x+3)^5(x^5 - 2)^3$

   (b) $y = x^{sin\ x}$

2 Exponential Growth and Decay

Recall that for population growth ($k > 0$) or for natural decay ($k < 0$) we have

$$\frac{dy}{dt} = ky$$

We also know that a solution is

$$y = Ae^{kt}$$

Where A is a constant. Since $y(0) = A \implies A$ is the *initial value*

Thus a solution to the *initial value problem* (IVP) $\frac{dy}{dt} = ky \;,\; y(0) = y_0$ is $y(t) = y_0 e^{kt}$

ex 6 A bacteria divides into 2 cells every 20 minutes. If the initial population is 60 cells

1. Find the growth rate.

$$P(t) = 60e^{kt} \implies P(1/3) = 60e^{k(1/3)} \implies k = ln\ 8$$

2. Find an expression for the number of cells after t hours.

$$P(t) = 60e^{(ln\ 8)t} = 60 \cdot 8^t$$

3. Find the number of cells after 8 hours.

$$P(8) = 1,006,632,960$$

4. Find the **rate of growth** after 8 hours.

$$\frac{dP}{dt} = kP \implies P'(8) = kP(8) = ln\,8{\cdot}P(8) \;\approx\; 2.01\,billion\,cells/hour$$

5. When will the population reach 20,000?

$$60 \cdot 8^t = 20,000 \quad\implies\quad t \;\approx\; 2.8\;hours$$

For radioactive decay

$$\frac{dm}{dt} \;=\; km \quad so \quad m(t) = m_0 e^{kt}$$

ex 7 Bismuth-210 has a half-life of 5 days. If a sample starts with 800 mg

1. Find a formula for the mass remaining after t days.

$$y(t) = 800e^{kt}\;,\; y(5) = 400 \;\implies\; k = -\frac{ln\;2}{5} \;\implies\; y(t) = 800{\cdot}2^{-t/5}$$

2. Find the mass after 30 days.

$$y(30) = 12.5\;mg$$

3. When is the mass 1 mg?

$$1 = 800 \cdot 2^{-t/5} \quad\implies\quad t \;\approx\; 48\;days$$

Newton's Law of Cooling

$$\frac{dT}{dt} = k(T(t) - T_s)$$

where $T(t)$ is the temperature of the object at time t and T_s is the temperature of the surroundings.

Since T_s is a constant, let $y(t) = T(t) - T_s \implies y'(t) = T'(t)$ thus

$$\frac{dy}{dt} = ky$$

The idea here is to substitute $y(t) = T(t) - T_s$ since that reduces the problem to one we have dealt with already. In other words turn Newton's Law of Cooling into a population growth problem. We do this since we already have a general solution for the population growth problem. Let's try an example.

ex 8 A thermometer is taken from a 20° C room outside where it is 5° C. After one minute the thermometer reads 12° C

1. What will the reading be after 2 minutes?

   $$\frac{dT}{dt} = k(T(t)-5)\ \ so\,let\ \ y(t) = T(t)-5 \implies y(t) = y_0e^{kt} = 15e^{kt}$$
   $$so\ \ T(t) = 5+15e^{kt}\ and\,T(1) = 12 = 5+15e^{kt} \implies k = ln(7/15) \implies$$
   $$T(2) \approx 8.3^\circ\ C$$

2. When will the thermometer read 6° C?

   $$5 + 15e^{ln(7/15)t} = 6 \iff t \approx 3.6\ minutes$$

Worksheet for Section 2

1. A bacteria culture grows with a constant relative growth rate. After 2 hours there are 600 and after 8 hours there are 75,000.
   (a) Find the initial population.
   (b) Find a formula for the population after t hours.
   (c) Find the number of bacteria after 5 hours.
   (d) Find the rate of growth after 5 hours.
   (e) When will the population be 200,000?
2. After 3 days a sample of radon-222 decayed to 58% of its original amount.
   (a) What is its half-life?
   (b) How long until the sample decays to 10% of its original amount

Homework for Section 2

1. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After one hour the population is 420.

   (a) Find an expression for the number of bacteria after t hours.

   (b) Find the number of bacteria after 3 hours.

   (c) Find the rate of growth after 3 hours.

   (d) When will the population reach 10,000?

2. The half-life of Cesium-137 is 30 years. Suppose we have a 100 mg sample

   (a) Find an expression for the mass after t years.

   (b) How much remains after 100 years?

   (c) When will the mass be 1 mg?

3. A turkey is taken from the oven when its temperature has reached 185°F and placed in a room where the temperature is 75°F

   (a) If the temp is 150°F after 30 minutes, what is the temp after 45 minutes?

   (b) When will it have cooled to 100°F?

4. A cold drink is removed from the fridge and its temp is 5°C. After 25 minutes in a 20°C room its temp has increased to 10°C.

   (a) What is its temp after 50 minutes?

   (b) When will its temp be 15°C?

3 Related Rates

The idea here is to compute the rate of change of one quantity in terms of the rate of change of another

Then relate the two quantities and use the chain rule to differentiate **with respect to time** t

ex 9 If $x^2 + y^2 = 25$ and $\frac{dy}{dt} = 6$, find $\frac{dx}{dt}$ when $y = 4$

$$x^2 + y^2 = 25 \iff 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies \frac{dx}{dt} = \frac{-2y\frac{dy}{dt}}{2x}$$

When $y = 4 \implies x = 3 \ \ since \ \ x^2 + y^2 = 25$

$$\implies \frac{dx}{dt} = \frac{(-2)(4)(6)}{(2)(3)} = -8$$

ex 10 Air is being pumped into a spherical balloon so that the volume increases at a rate of 100 cm^3/s. How fast is the radius increasing when the diameter is 50 cm?

What information is given?
Volume increases *at a rate of*

$$\implies \frac{dV}{dt} = 100$$

$$Find \ \ \frac{dr}{dt} \ \ when \ \ r = 25$$

So we need to relate V and r ...

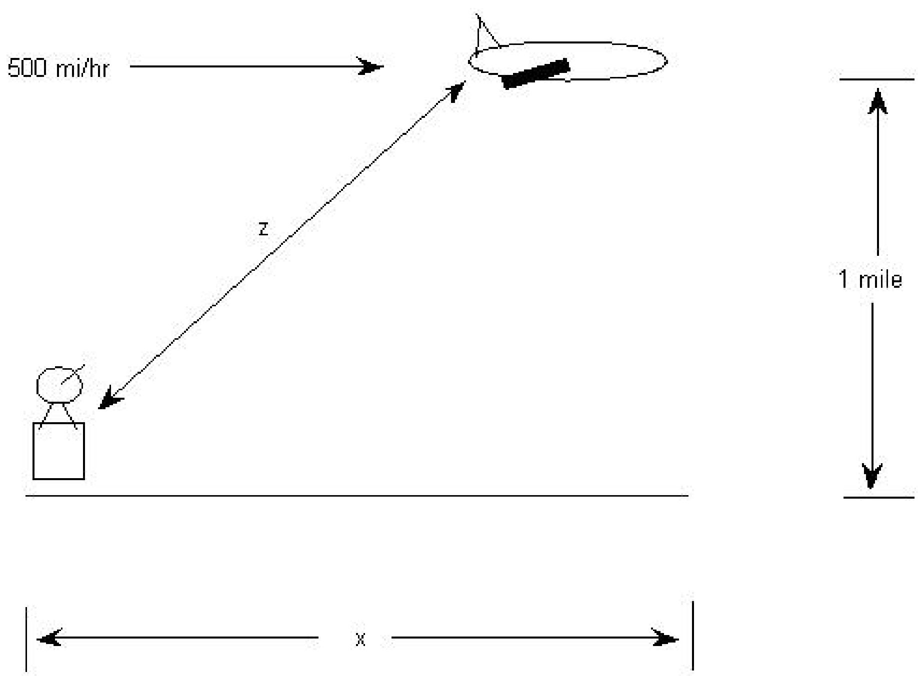
$$V = \frac{4}{3}\pi r^3$$

Now differentiate with respect to time t

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{100}{4\pi r^2} = \frac{100}{4\pi (25)^2} = \frac{1}{25\pi}\ cm/s$$

ex 11 A plane is flying horizontally at an altitude of 1 mile and a speed of 500 mi/hr as it passes over a radar station. Find the rate at which the distance from the plane to the station is increasing when the plane is 2 miles away from the station.

So we get something like this ...



$$\frac{dx}{dt} = 500, \quad find\ \frac{dz}{dt}\ \ when\ \ z = 2$$

So we need to relate x and z

$$x^2 + 1^2 \; = \; z^2 \quad \Longleftrightarrow \quad 2x\frac{dx}{dt} \; = \; 2z\frac{dz}{dt} \quad \Longleftrightarrow \quad \frac{dz}{dt} \; = \; \frac{x\frac{dx}{dt}}{z}$$

When $z = 2 \implies x = \sqrt{3}$ so

$$\frac{dz}{dt} \; = \; \frac{\sqrt{3}(500)}{2} \; = \; 250\sqrt{3} \; mi/hr$$

Worksheet for Section 3

1. As a circular metal grate is being heated, its diameter changes at a rate of .01 cm/min. Find the rate at which the area of one side is changing when the diameter is 30 cm.

2. Suppose a spherical snowball is melting and the radius is decreasing at a constant rate, changing from 12 in to 8 in in 45 min. How fast was the volume changing when the radius was 10 in?

3. A ladder 20 ft long leans against a vertical wall. If the bottom of the ladder slides away from the building at a rate of 2 ft/sec, at what rate is the angle between the ladder and the ground changing when the top of the ladder is 12 ft above the ground?

4. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/sec. How fast is the distance from the particle to the origin changing at this instant?

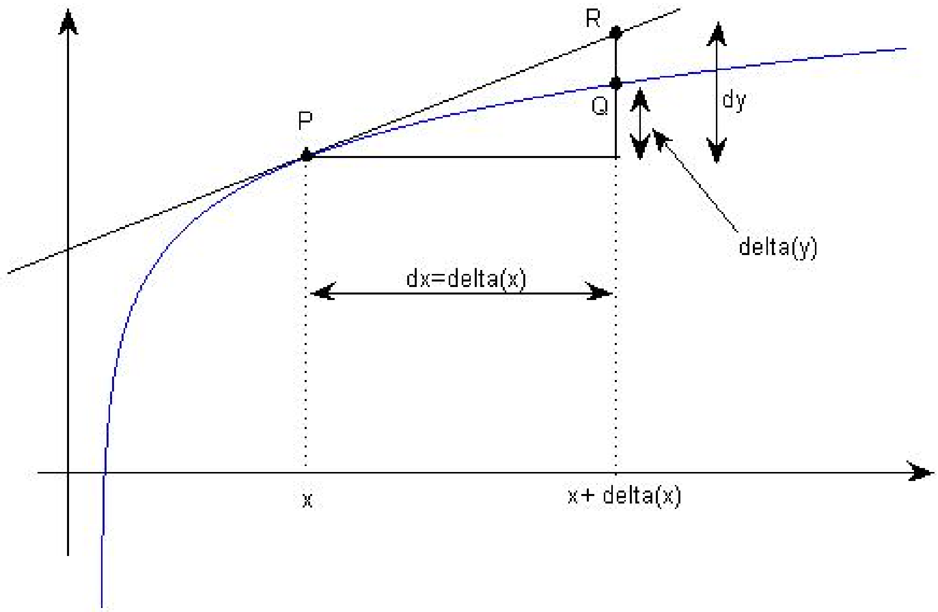
5. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of .06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.

6. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4:00 P.M.?

Homework for Section 3

1. If $y = x^3 + 2x$ and $dx/dt = 5$, find dy/dt when $x = 2$
2. If $z^2 = x^2 + y^2$, $dx/dt = 2$ and $dy/dt = 3$, find dz/dt when $x = 5$ and $y = 12$
3. A street light is mounted at the top of a 15ft pole. A man 6 feet tall walks away from the pole at a rate of 5 ft/sec in a straight line. How fast is the tip of his shadow moving when he is 40 ft from the pole?
4. At noon, ship A is 150km west of ship B. Ship A is sailing east at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4 PM?
5. Two cars start moving from the same point. One travels south at 60 mi/hr and the other west at 25 mi/hr. At what rate is the distance between the cars increasing two hours later?

4 Linear Approximations



Take a look at the tangent line PR above. At the point P, does the tangent line approximate the curve well?

This is the basic idea of this section, that a tangent line, which is easy to find, can approximate the curve *very* well near the point it touches of course. The further away you are from the tangent point, the worse the approximation.

How do we find the equation of the tangent line?

$$y - f(a) = f'(a)(x - a)$$

$$or$$

$$y = f(a) + f'(a)(x - a)$$

So a *linear approximation* or a *tangent line approximation* of f at a is:

$$f(x) \approx f(a) + f'(a)(x - a)$$

The linear function whose graph is the tangent line

$$L(x) = f(a) + f'(a)(x - a)$$

is called the *linearization* of f at a

This is also useful to predict future behavior given past data points.

ex 12 Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$ Are these approximations over estimates or under estimates? Why?

So, since $a = 1$, we have that $f(x) = \sqrt{x+3} \implies f(a) = 2$ and

$$f'(x) = \frac{1}{2\sqrt{x+3}} \implies f'(a) = \frac{1}{4}$$

Thus

$$L(x) = f(a) + f'(a)(x - a) = 2 + \frac{1}{4}(x - 1) = \frac{7 + x}{4}$$

In other words, $L(x)$ approximates $\sqrt{x+3}$ very well **near 1**. So

$$\sqrt{3.98} = \sqrt{.98 + 3} \approx \frac{7 + .98}{4} = 1.995$$

and

$$\sqrt{4.05} \;=\; \sqrt{1.05+3} \;\approx\; \frac{7+1.05}{4} \;=\; 2.0125$$

If you plug $\sqrt{3.98}$ and $\sqrt{4.05}$ into your calculator you will see that these are excellent approximations because *1.05 and .98 are very close to 1*

Why do you think both of these are over estimates?

ex 13 Find the linearization of $f(x) = sin\ x$ at $a = 0$

$$f(x) = sin\ x \;\;\Longrightarrow\;\; f'(x) = cos\ x \;\; so$$

$$L(x) \;=\; f(a) + f'(a)(x-a) \;=\; sin\ 0 + cos\ 0(x-0) \;=\; x$$

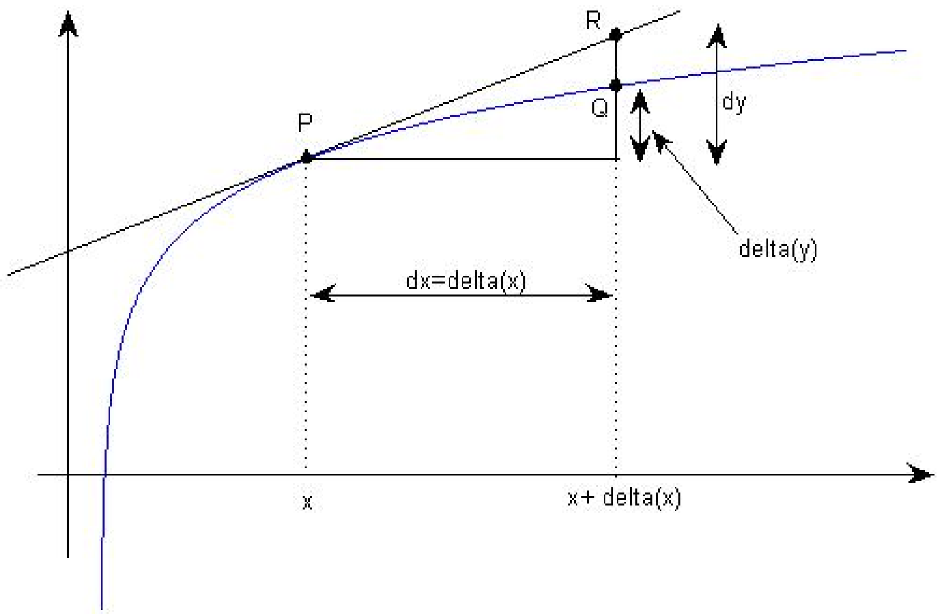
Thus

$$sin\ x \;\approx\; x \quad (NEAR\ 0)$$

How do you think this might be useful?

Differentials

These are the exact same idea as linear approximation.



Here dy represents the amount the *tangent line* rises or falls

Δy represents the amount the *curve* rises or falls

$$\begin{aligned} f'(x) &= slope = \frac{dy}{dx} \implies dy = f'(x)dx \\ \Delta y &= f(x+\Delta x) - f(x) \\ dx &= \Delta x \end{aligned}$$

dy is called the *differential*

You can see the idea is the same as linear approximation by observing that

$$dy = f'(x)dx = f'(x)\Delta x = f'(x)(x-a)$$

which is part of $L(x)$

The only difference between dy and $L(x)$ is that $L(x)$ adds back the amount $f(a)$

ex 14 Compare values of Δy and dy if $f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.05

$$f(2) = 9, \quad f(2.05) = 9.717625 \quad \Longrightarrow \quad \Delta y \ = \ .717625$$

So the actual function rises .717625 exactly from $x = 2$ to $x = 2.05$

Differentials are basically approximations so for this example

$$dy \ = \ f'(x)dx \ = \ (3x^2 + 2x - 2)dx$$

in general and when $x = 2$ and $\Delta x = .05 \quad \Longrightarrow \quad dy \ = \ (3(2)^2 + 2(2) - 2)(.05) \ = \ .7$

You can see that dy approximates Δy fairly well. It gets better as Δx becomes smaller.

The idea is that in general, dy is sometimes easier to compute than Δy

Recall from example 29 that $f(x) = \sqrt{x+3}$ so

$$dy \ = \ f'(x)dx \ = \ \frac{dx}{2\sqrt{x+3}}$$

if $a = 1$ and $\Delta x = dx = .05$ then

$$dy \ = \ \frac{.05}{2\sqrt{1+3}} \ = \ .0125 \ \ and \ \ \sqrt{4.05} \ = \ f(1.05) \ \approx \ f(1) + dy \ = \ 2.0125$$

as before

Worksheet for Section 4

1. Find the linearization, $L(x)$, of $f(x) = ln\ x$ at $a = 1$.
2. Find and evaluate dy if $y = 1/(x+1)$, $x = 1$, $dx = -0.01$.

Homework for Section 4

1. Find the linearization, $L(x)$, of the function at a.

   (a) $f(x) = x^4 + 3x^2$, $a = -1$

   (b) $f(x) = cos\ x$, $a = \pi/2$

2. Find the differential for $y = x^2 sin\ 2x$

3. Find the differential dy and evaluate it for the given values of x and dx

   (a) $y = e^{x/10}$, $x = 0$, $dx = 0.1$

   (b) $y = 1/(x+1)$, $x = 1$, $dx = -0.01$

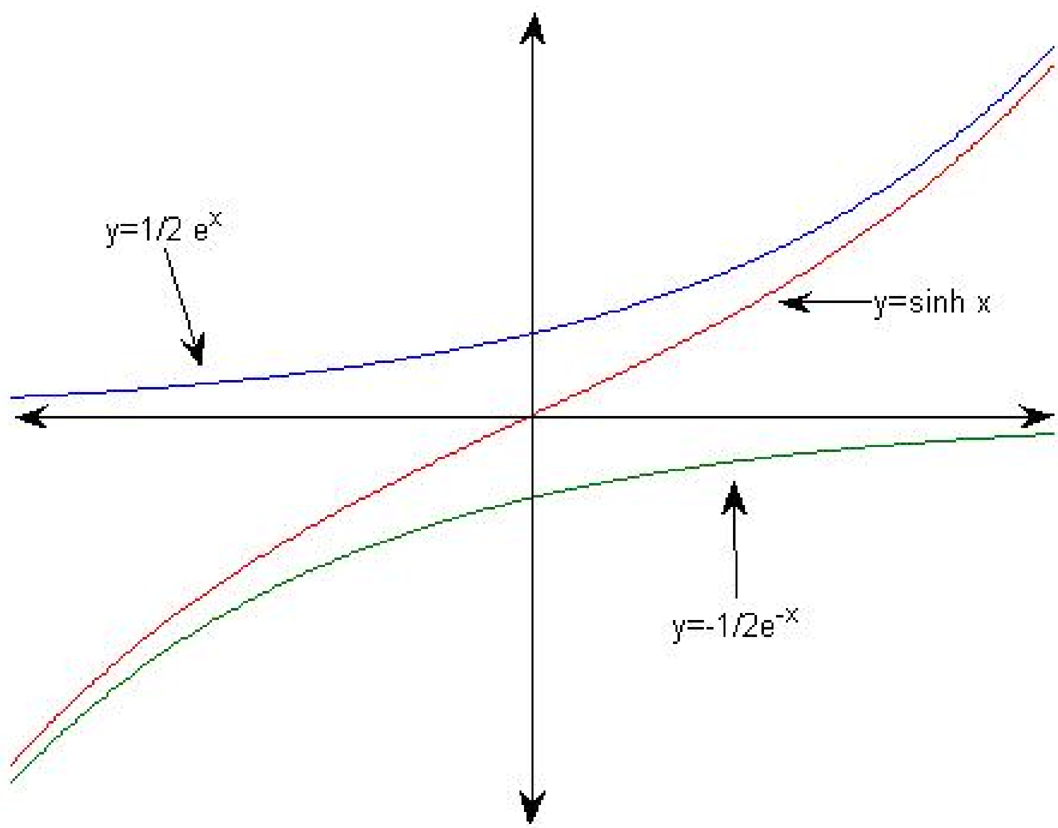
4. Use a linear approximation to estimate $(2.001)^5$

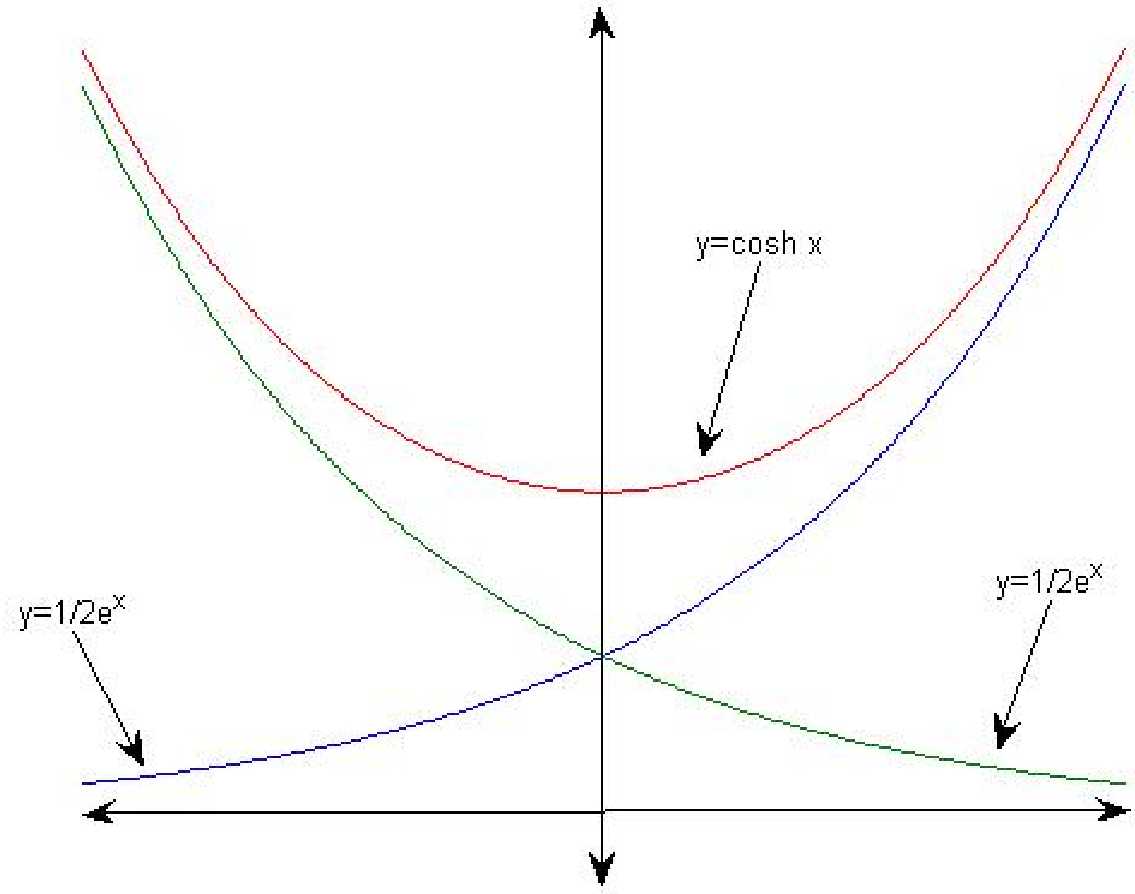
5 Hyperbolic Functions

Certain combinations of e^x and e^{-x} arise frequently in engineering so we call them the Hyperbolic Functions

$$sinh\ x \ = \ \frac{e^x - e^{-x}}{2} \qquad and \qquad cosh\ x \ = \ \frac{e^x + e^{-x}}{2}$$

We use hyperbolic sine and hyperbolic cosine because these have the same relationship to the hyperbola that the trig functions have to the circle. It is a *descriptive* name





If you examine the graphs of $y = cosh\ x$ and $y = sinh\ x$ can you think of what they might model?

$sinh\ x$ represents a gradual absorption or decay of light, heat, radioactivity, etc...
$cosh\ x$ represents a catenary. Think about how an electrical wire bends between two poles.

How about the derivatives? Since they are combinations of exponentials they are easy to compute

$$\frac{d}{dx}(sinh\ x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = cosh\ x$$

likewise we find that

$$\frac{d}{dx}(cosh\ x) \ = \ sinh\ x$$

You should be able to figure out the others but we will focus on the big two

ex 15 Find $f'(x)$ if $f(x) = sinh^2\ x$

$$sinh^2\ x \ = \ (sinh\ x)^2 \ \implies \ f'(x) \ = \ 2(sinh\ x)(cosh\ x)$$

ex 16 Find $g'(x)$ if $g(x) = ln(cosh\ x)$

$$g'(x) \ = \ \frac{\frac{d}{dx}(cosh\ x)}{cosh\ x} \ = \ \frac{sinh\ x}{cosh\ x} \ = \ tanh\ x$$

Worksheet for Section 5

1. Prove the following identity: $cosh\ x + sinh\ x = e^x$
2. Find the derivative of $y = sinh(cosh\ x)$
3. Find the derivative of

$$y = e^{cosh\ 3x}$$

Homework for Section 5

1. Prove the following: $sinh(-x) = -sinh\ x$

2. Differentiate the following:

   (a) $f(x) = tanh(1 + e^{2x})$

   (b) $f(x) = xsinh\ x - cosh\ x$

   (c) $f(x) = cosh(ln\ x)$

   (d) $f(x) = ln(cosh\ x)$