MATH 151 Worksheet Keys Exam 5

Worksheet for Section 1

1. Let $x = 1 + t$, $y = 5 - 2t$ and $-2 \leq t \leq 3$:

   (a) Sketch the curve and indicate direction as t increases.

t	-2	-1	0	1	2	3
x	-1	0	1	2	3	4
y	9	7	5	3	1	-1

   (b) Eliminate the parameter to find a Cartesian equation of the curve.

$$x = 1 + t \quad \Longrightarrow \quad t = x - 1$$

$$\Longrightarrow \quad y = 5 - 2(x - 1) = -2x + 7$$

$$\text{so} \;\; y = -2x + 7 \;\; \text{for} \;\; -1 \leq x \leq 4$$

2. Describe the motion of the particle given by $x = 2 + cos\ t$, $y = 3 + sin\ t$, $0 \leq t \leq 2\pi$

$$x - 2 = cos\ t\ ,\ y - 3 = sin\ t \quad \Longrightarrow \quad sin^2\ t + cos^2\ t = (y-3)^2 + (x-2)^2 = 1$$

the motion is on the unit circle centered at $(2, 3)$ as t goes from 0 to 2π

the particle makes one complete counterclockwise rotation starting and ending at $(3, 3)$

Worksheet for Section 2

1. Find an equation of the tangent to $x = t^4 + 1$, $y = t^3 + t$ at the point corresponding to $t = -1$.

$$\frac{dy}{dt} = 3t^2 + 1 \ , \ \frac{dx}{dt} = 4t^3 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{3t^2+1}{4t^3}$$

$$t = -1 \quad \Longrightarrow \quad (x, y) = (2, -2) \text{ so } \frac{dy}{dx} = \frac{4}{-4} = -1$$

$$\text{the equation is } y - (-2) = (-1)(x - 2) \quad \Longrightarrow \quad y = -x$$

2. Find the length of the curve $x = e^t \; cos \; t, \quad y = e^t \; sin \; t, \quad 0 \le t \le \pi$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(e^t(cos\; t - sin\; t)\right)^2 + \left(e^t(cos\; t + sin\; t)\right)^2 = 2e^{2t}$$

$$\Longrightarrow \quad L = \int_0^\pi \sqrt{2e^{2t}} \; dt = \int_0^\pi \sqrt{2}e^t \; dt = \sqrt{2}(e^\pi - 1)$$

3. Find the area of the surface obtained by rotating $x = 3t - t^3, \quad y = 3t^2, \quad 0 \le t \le 1$ about the $x - axis$.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(3 - 3t^2\right)^2 + \left(6t\right)^2 = (3(1 + t^2))^2$$

$$\Longrightarrow \quad SA = \int_0^1 (2\pi)(3t^2)(3)(1 + t^2) \; dt = 18\pi \int_0^1 (t^2 + t^4) \; dt = \frac{48}{5}\pi$$

Worksheet for Section 3

1. Identify the curve by finding a Cartesian equation for $r = 2\ sin\ \theta + 2\ cos\ \theta$

$$r = 2\ sin\ \theta + 2\ cos\ \theta \implies r^2 = 2r\ sin\ \theta + 2r\ cos\ \theta \implies x^2 + y^2 = 2x + 2y$$

$$\implies (x-1)^2 + (y-1)^2 = 2 \implies \text{a circle}$$

2. Find a polar equation represented by the Cartesian equation $x^2 + y^2 = 9$

$$x^2 + y^2 = 9 \implies r^2 = 9 \implies r = 3 \text{ or } r = -3$$

3. Find the points on the curve $r = e^\theta$ where the tangent line is horizontal or vertical.

$$\frac{dy}{d\theta} = e^\theta sin\ \theta + e^\theta cos\ \theta = 0 \implies sin\ \theta = -cos\ \theta \implies tan\ \theta = -1$$

$$\text{so } \theta = -\frac{\pi}{4} + n\pi \quad (n \in \mathbb{Z}) \text{ horizontal tangents at } \left(e^{-\pi/4+n\pi}, -\frac{\pi}{4} + n\pi\right)$$

$$\frac{dx}{d\theta} = e^\theta cos\ \theta - e^\theta sin\ \theta = 0 \implies sin\ \theta = cos\ \theta \implies tan\ \theta = 1$$

$$\text{so } \theta = \frac{\pi}{4} + n\pi \quad (n \in \mathbb{Z}) \text{ vertical tangents at } \left(e^{\pi/4+n\pi}, \frac{\pi}{4} + n\pi\right)$$

Worksheet for Section 4

1. Sketch the curve $r = 3\ cos\ \theta$ and find the area in encloses.

this is a circle that sits against the "y" axis from (0,0) to (3,0)

$$A = 2\int_0^{\pi/2} \frac{1}{2}r^2\ d\theta = \int_0^{\pi/2} (3cos\ \theta)^2\ d\theta = 9\int_0^{\pi/2} cos^2\ \theta\ d\theta$$

$$= 9\int_0^{\pi/2} \left(\frac{1}{2} + \frac{cos\ 2\theta}{2}\right)\ d\theta = \frac{9\pi}{4}$$

2. Find the area enclosed by one loop of $r = 3\ cos\ 5\theta$

$$r = 0 \implies 3cos\ 5\theta = 0 \implies 5\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{10}$$

$$A = 2\int_0^{\pi/10} \frac{1}{2}(3cos\ 5\theta)^2\ d\theta = \frac{9}{2}\int_0^{\pi/10} (1 + cos\ 10\theta)\ d\theta = \frac{9\pi}{20}$$

3. Find the length of the curve $r = e^{2\theta}$, $0 \le \theta \le 2\pi$

$$L = \int_0^{2\pi} \sqrt{(e^{2\theta})^2 + (2e^{2\theta})^2}\ d\theta = \int_0^{2\pi} \sqrt{5e^{4\theta}}\ d\theta$$

$$= \sqrt{5}\int_0^{2\pi} e^{2\theta}\ d\theta = \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)$$