1 Improper Integrals

The Fundamental Theorem of Calculus assumes two things, what are they?

1. The definite integral must have a finite interval, that is on $[a, b]$ and
2. The integrand, $f(x)$ must be continuous

Violating either of these results in what is called an *Improper Integral*

There are 2 types:

1.1 TYPE I: Infinite Intervals

ex 1 Consider the following where $b > 1$

$$\int_1^b \frac{1}{x^2}\ dx$$

$$= -\frac{1}{x}\Bigg|_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$$

note that no matter how large b is, the area under the curve is always less than 1

that is

$$\textit{as } b \longrightarrow \infty \qquad \int_1^\infty \frac{1}{x^2}\ dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\ dx = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$$

Definition of Type I

1. If $\int_a^t f(x)\,dx$ exists $\forall\ t \geq a$, then

$$\int_a^{\infty} f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$$

provided the limit exists ***as a finite number***

2. If $\int_t^b f(x)\,dx$ exists $\forall\ t \leq b$, then

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx$$

provided the limit exists ***as a finite number***

The *Improper Integrals*

$$\int_a^{\infty} f(x)\ dx \ \ and \ \ \int_{-\infty}^{b} f(x)\ dx$$

are called **convergent**, or what I will write as **C** if the corresponding limit exists and **divergent**, or **D** if the limit DNE.

3. If both

$$\int_a^{\infty} f(x)\ dx \ \ and \ \ \int_{-\infty}^{a} f(x)\ dx$$

are convergent then

$$\int_{-\infty}^{\infty} f(x)\ dx \ = \ \int_{-\infty}^{a} f(x)\ dx \ + \ \int_a^{\infty} f(x)\ dx$$

Any real number a can be used

ex 2 Find

$$\int_1^\infty \frac{1}{x}\ dx$$

$$= \ \ldots \ = \lim_{t\to\infty}\ ln\ \mid t \mid -ln\ 1 = ln\ \infty = \infty$$

thus this integral **diverges**

ex 3 Find

$$\int_0^\infty e^{-x}\ dx$$

$$= \ \ldots \ = \lim_{t\to\infty}\ (-e^{-t}+1) = 1$$

thus this integral **converges** and is 1

ex 4 Find

$$\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\ dx$$

$$= \ \ldots \ = \lim_{t\to-\infty}\ (tan^{-1}\ e^x)\Big|_b^0 + \lim_{t\to\infty}(tan^{-1}\ e^x)\Big|_0^t = \ldots = \frac{\pi}{2}$$

Note that:

$$\int_1^\infty \frac{1}{x^p}\ dx = \begin{cases} \frac{1}{p-1}, & if\ p > 1\ (thus\ \mathbf{C}) \\ \mathbf{D}, & if\ p \leq 1 \end{cases}$$

1.2 TYPE II: Discontinuous Integrands

Definition of Type II

1. If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\ dx = \lim_{t \to b^-} \int_a^t f(x)\ dx$$

   provided the limit exists ***as a finite number***

2. If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\ dx = \lim_{t \to a^+} \int_t^b f(x)\ dx$$

   provided the limit exists ***as a finite number***

3. If f has a discontinuity at c, where $a < c < b$, then

$$\int_a^b f(x)\ dx \ = \ (1)\ \int_a^c f(x)\ dx \ + \ (2)\ \int_c^b f(x)\ dx$$

provided **both** (1) and (2) are convergent

ex 5 Find

$$\int_{-1}^{2} \frac{1}{x^3}\ dx$$

- Note that we have a discontinuity at $x = 0$

$$\int_{-1}^{2} \frac{1}{x^3}\ dx \ = \ \int_{-1}^{0} \frac{1}{x^3}\ dx \ + \ \int_{0}^{2} \frac{1}{x^3}\ dx$$

Now

$$\int_0^2 \frac{1}{x^3}\,dx = \lim_{t\to 0^+} \int_t^2 \frac{1}{x^3}\,dx = \lim_{t\to 0^+} \left(-\frac{1}{2x^2}\right)\Bigg|_t^2 = \ldots = \infty$$

Thus the integral **diverges**

It is important to realize that:

$$\int_{-1}^2 \frac{1}{x^3}\,dx \neq -\frac{1}{2x^2}\Bigg|_{-1}^2 = \frac{3}{8}$$

Sometimes it is impossible to find the exact value of the integral but you would still like to know if it converges or diverges.

Comparison Theorem

Suppose that f and g are continuous and $f \geq g \geq 0$ for $x \geq a$, then:

1. If $\displaystyle\int_a^\infty f$ is convergent $\implies$ $\displaystyle\int_a^\infty g$ is convergent
2. If $\displaystyle\int_a^\infty g$ is divergent $\implies$ $\displaystyle\int_a^\infty f$ is divergent

- This is an important concept that we will return to later •

ex 6 Is the following integral **C** or **D**, that is, convergent or divergent?

$$\int_1^\infty \frac{1+e^{-x}}{x}\ dx$$

Here we will need the Comparison Theorem since the integrand has no antiderivative

Since

$$\frac{1+e^{-x}}{x} > \frac{1}{x} \quad \Longrightarrow \quad \mathbf{D}$$

Worksheet for Section 1

1. Evaluate the following improper integrals:

   (a) $\displaystyle\int_1^{\infty} \frac{1}{(3x+1)^2}\ dx$

   (b) $\displaystyle\int_{-2}^{3} \frac{1}{x^4}\ dx$

Homework for Section 1

1. Determine convergence or divergence. Evaluate those that converge.

   (a) $\displaystyle\int_1^\infty \frac{1}{(3x+1)^2}\,dx$

   (b) $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt{2-x}}\,dx$

   (c) $\displaystyle\int_4^\infty e^{-x/2}\,dx$

   (d) $\displaystyle\int_{-\infty}^\infty \frac{x}{1+x^2}\,dx$

   (e) $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

   (f) $\displaystyle\int_{-2}^3 \frac{1}{x^4}\,dx$

2. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

   (a) $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

   (b) $\displaystyle\int_0^\infty \frac{arctan\ x}{2+e^x}\,dx$

2 Indeterminate Forms and L'Hospital's Rule

Let's try to evaluate the following limit

$$\lim_{x \to 1} \frac{ln\ x}{x-1}$$

Right now the only way is to approximate. We know that we can not plug in 1 since that would give us $\frac{0}{0}$. We will now discuss a new approach on how to deal with limits of this type.

If you have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \textit{where BOTH } f(x) \longrightarrow 0 \textit{ and } g(x) \longrightarrow 0 \textit{ when } x \to a$$

then the limit may or may not exist and is called an INDETERMINATE FORM of type $\frac{0}{0}$

Recall that:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \ \ldots \ \textit{ here we can cancel}$$

$$\lim_{x \to 0} \frac{sin\ x}{x} = 1 \ \textit{ by a geometric argument}$$

These methods do not work on

$$\lim_{x \to 1} \frac{ln\ x}{x-1}$$

This gives us

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near a. If

$$\lim_{x \to a} g(x) = 0 \quad and \quad \lim_{x \to a} f(x) = 0$$

$$OR\ THAT$$

$$\lim_{x \to a} g(x) = \pm\infty \quad and \quad \lim_{x \to a} f(x) = \pm\infty$$

$$Then\ you\ have\ an\ indeterminate\ form\ of\ type\ \tfrac{0}{0}\ or\ \tfrac{\pm\infty}{\pm\infty}$$

$$THUS \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Provided the RHS exists or is $\pm\infty$

ex 7 Find

$$\lim_{x \to 1} \frac{ln\ x}{x - 1}$$

$$\lim_{x \to 1} \frac{ln\ x}{x - 1} = \frac{0}{0} \Longrightarrow \stackrel{LH}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

Note that you DO NOT use the quotient rule with L'Hospital's Rule!

ex 8 Find

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty} \implies \overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$$

Now what? Do it again, as it still qualifies as an indeterminate form.

$$\implies \overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

ex 9 Find

$$\lim_{x\to\pi^-} \frac{sin\ x}{1 - cos\ x}$$

$$\lim_{x\to\pi^-} \frac{sin\ x}{1 - cos\ x} = \frac{0}{1-(-1)} = 0$$

Be careful, L'Hospital's Rule does NOT apply here.

There are some other Indeterminate Forms as well:

Suppose $\lim_{x\to a}\ f(x) = 0$ and $\lim_{x\to a}\ g(x) = \pm\infty$

then $\lim_{x\to a}\ f(x)\cdot g(x)\ =\ ?$ This is an indeterminate form of type $0\cdot\infty$

To solve these you simply rewrite. Let

$$fg \ = \ \frac{f}{\frac{1}{g}} \ \ or \ \ \frac{g}{\frac{1}{f}}$$

This will turn the limit into type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

ex 10 Find

$$\lim_{x \to 0^+} \ x \ ln \ x$$

$$\lim_{x \to 0^+} \ x \ ln \ x \ = \ (0)(-\infty) \ = \ \lim_{x \to 0^+} \ \frac{ln \ x}{\frac{1}{x}} \ = \ \frac{-\infty}{\infty}$$

$$\stackrel{LH}{=} \ \lim_{x \to 0^+} \ \frac{\frac{1}{x}}{\frac{-1}{x^2}} \ = \ \lim_{x \to 0^+} \ -x \ = \ 0$$

Worksheet for Section 2

MATH 151 **Section 2 Worksheet (A)** **Exam 2**

1.
$$\lim_{\theta \to \frac{\pi}{2}} \frac{1 - sin\ \theta}{csc\ \theta}$$

2.
$$\lim_{t \to 0} \frac{e^{3t} - 1}{t}$$

3.
$$\lim_{x \to \infty} \frac{ln\ x}{x}$$

4.
$$\lim_{x \to 0} \frac{tan\ px}{tan\ qx}$$

Another Indeterminate Form:

Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$

then $\lim_{x\to a} [f(x) - g(x)] = ?$ This is an indeterminate form of type $\infty - \infty$

To solve these you convert the difference into a quotient, rationalize or factor.

ex 11 Find

$$\lim_{x\to(\pi/2)^-} (sec\ x - tan\ x)$$

$$\lim_{x\to(\pi/2)^-} (sec\ x - tan\ x) = (\infty - \infty) \implies \lim_{x\to(\pi/2)^-} (sec\ x - tan\ x)$$

$$= \lim_{x\to(\pi/2)^-} \left(\frac{1}{cos\ x} - \frac{sin\ x}{cos\ x}\right)$$

$$= \lim_{x\to(\pi/2)^-} \left(\frac{1 - sin\ x}{cos\ x}\right) = \frac{0}{0} \implies \overset{LH}{=} \lim_{x\to(\pi/2)^-} \left(\frac{-cos\ x}{-sin\ x}\right) = 0$$

Indeterminate Powers

these are of the form:

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. type 0^0
2. type ∞^0
3. type 1^∞

These can be solved using the properties of logs. If

$$y = [f(x)]^{g(x)} \implies ln\ y = g(x)\ ln\ f(x) \implies$$
$$\text{indeterminate form of type } 0 \cdot \infty$$

ex 12 Find

$$\lim_{x \to 0^+} x^x$$

$$Let\ \ y = x^x \implies ln\ y = x\ ln\ x \implies \lim_{x \to 0^+} ln\ y = \lim_{x \to 0^+} x\ ln\ x$$
$$Since\ \lim_{x \to 0^+} x\ ln\ x = 0 \text{ from ex 4} \implies \lim_{x \to 0^+} ln\ y = 0 \implies$$
$$\lim_{x \to 0^+} y = e^0 = 1$$

1.
$$\lim_{x\to 0} \quad (cos\ 3x)^{\frac{5}{x}}$$

2.
$$\lim_{x\to -1} \quad \frac{x^2-1}{x+1}$$

3.
$$\lim_{x\to 0} \quad \frac{e^x-1-x}{x^2}$$

4.
$$\lim_{x\to \infty} \quad \left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x$$

5.
$$\lim_{x\to \infty} \quad (e^x+x)^{\frac{1}{x}}$$

6.
$$\lim_{x\to \infty} \quad x\ tan\left(\frac{1}{x}\right)$$

Homework for Section 2

1. Find the following limits:

   (a)
   $$\lim_{x\to 1} \frac{x^9 - 1}{x^5 - 1}$$

   (b)
   $$\lim_{x\to \pi/2^+} \frac{cos\ x}{1 - sin\ x}$$

   (c)
   $$\lim_{t\to 0} \frac{e^t - 1}{t^3}$$

   (d)
   $$\lim_{x\to 0^+} \frac{ln\ x}{x}$$

   (e)
   $$\lim_{x\to \infty} \frac{e^x}{x^3}$$

   (f)
   $$\lim_{x\to 0} \frac{sin^{-1}\ x}{x}$$

   (g)
   $$\lim_{x\to \infty} x\ sin(\pi/x)$$

   (h)
   $$\lim_{x\to 1^+} ln\ x\ tan(\pi x/2)$$

   (i)
   $$\lim_{x\to \infty} (x - ln\ x)$$

3 Sequences

Definition:

A *sequence* is simply a function whose domain is the set of positive integers.

That is, a list in a definite order.

$$a_1, a_2, a_3, \ \ldots \ , a_n, \ \ldots$$

We will deal with *infinite sequences*

There are basically three ways to express a sequence

1. $$\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$$

2. $$a_n = \frac{n}{n+1}$$

3. $$\left(\frac{1}{2}, \frac{2}{3}, \ldots, \frac{n}{n+1}, \ldots\right)$$

There are some sequences that do not have defining equations.

Let a_n = the nth decimal place of the number e.

Then $a_n = \{7, 1, 8, 2, 8, 1, 8, 2, \ldots\}$

ex 13 What happens to the following as $n \longrightarrow \infty$

$$a_n = \frac{n}{n+1}$$

Well,

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

In general $\lim_{n\to\infty} a_n = L$ means what?

Definition:

$$\lim_{n\to\infty} a_n = L$$

if

$$\forall \ \epsilon > 0 \ \exists \ N \in \mathbb{N}$$

such that

$$n \ > \ N \implies | \, a_n - L \, | \ < \ \epsilon$$

We will discuss what this means in class.

- If the limit exists then the sequence ***converges***
- If the limit DNE then the sequence ***diverges***

Since we will be working with limits, let's quickly review the limit laws

Limit Laws

If $\{a_n\}$ and $\{b_n\}$ are convergent and c is a constant, then

1. $\lim_{n\to\infty} \ (a_n \pm b_n) \ = \ \lim_{n\to\infty} \ a_n \ \pm \ \lim_{n\to\infty} \ b_n$
2. $\lim_{n\to\infty} \ ca_n \ = \ c\lim_{n\to\infty} \ a_n$

3. $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$

4.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

provided

$$\lim_{n\to\infty} b_n \neq 0$$

5. $\lim_{n\to\infty} (a_n)^p = [\lim_{n\to\infty} a_n]^p$ if $p > 0$ and $a_n > 0$

Also the ***Squeeze Theorem*** applies

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

$$\Longrightarrow \lim_{n\to\infty} b_n = L$$

Theorem

If $\lim_{n\to\infty} | a_n | = 0$ then $\lim_{n\to\infty} a_n = 0$

For a proof think about the Squeeze Theorem

ex 14 Find

$$\lim_{n\to\infty} \frac{n}{1-2n}$$

- Recall that you can divide by the highest power in the denominator

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{n}-2} = -\frac{1}{2} \Longrightarrow \mathbf{C}$$

ex 15 Find

$$\lim_{n\to\infty} \frac{n^2}{2^n - 1}$$

- BE CAREFUL as L'Hospital's Rule does NOT apply to sequences, however...

Consider

$$\lim_{x\to\infty} \frac{x^2}{2^x - 1} = \dots = 0 \implies \mathbf{C}$$

Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ where n is an integer then $\lim_{n\to\infty} a_n = L$

ex 16 Is $a_n = 3 + (-1)^n$ convergent or divergent?

$$a_n = 2, 4, 2, 4, 2, 4, \dots \implies \mathbf{D}$$

The sequence $\{r^n\}$ is **convergent** if $-1 < r \leq 1$ and **divergent** for all other values of r

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & if \ -1 < r < 1 \\ 1, & if \ r = 1 \end{cases}$$

Definition

A sequence is *monotone* if its terms are nondecreasing OR nonincreasing, that is

$$a_1 \leq a_2 \leq \dots$$

or

$$a_1 \geq a_2 \geq \dots$$

Definition

A sequence a_n is *bounded above* if $\exists\ M \in\ \mathbb{R}$ such that $a_n \leq M \ \ \forall\ n$. The number M is called an *upper bound* of a_n

Definition

A sequence a_n is *bounded below* if $\exists\ m \in\ \mathbb{R}$ such that $a_n \geq m \ \ \forall\ n$. The number m is called an *lower bound* of a_n

Definition

A sequence a_n is *bounded* if it is bounded below OR bounded above. An important property of the real numbers is that they are *complete*. What this means informally is that there are no holes in the reals. This is NOT true for the rational numbers. Why?

We can use the completeness property to show that if a sequence has an upper bound then it must have what is called a *least upper bound* or *supremum*. That is, of all the possible upper bounds, one of them is the smallest.

Of course there is an analagous definition for *greatest lower bound* or *infimum*.

ex 17 Find the least upper bound of

$$a_n = \frac{n}{n+1}$$

Monotone Sequence Theorem

Every bounded, monotonic sequence is convergent.

Why do you think this is true?

Worksheet for Section 3

1. Determine whether the sequences converge or diverge. If it converges, find the limit.

   (a)

$$a_n = \frac{2^n}{3^{n+1}}$$

   (b)

$$a_n = \frac{(-1)^{n-1} n}{n^2 + 1}$$

Homework for Section 3

1. Determine if the following sequences converge or diverge. If the converge, find the limit.

   (a)
   $$a_n = \frac{n^3}{n^3 + 1}$$

   (b)
   $$a_n = \frac{3^{n+2}}{5^n}$$

   (c)
   $$a_n = \frac{(-1)^{n-1}}{n^2 + 1}$$

   (d)
   $$a_n = cos(n/2)$$

   (e)
   $$a_n = cos(2/n)$$

   (f)
   $$\{arctan\ 2n\}$$

   (g)
   $$\{n^2 e^{-n}\}$$

   (h)
   $$a_n = n\ sin(1/n)$$