MATH 254 Worksheets Part 1

Worksheet for Section 1

1. Solve the system using elementary row operations:

$$\begin{aligned} 2x_1 + 4x_2 &= -4 \\ 5x_1 + 7x_2 &= 11 \end{aligned}$$

2. Solve the system:

$$\begin{aligned} x_1 - 3x_2 \phantom{{}+5x_3} &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

3. Do the three planes $x_1 + 2x_2 + x_3 = 4$, $x_2 - x_3 = 1$ and $x_1 + 3x_2 = 0$ have at least one common point of intersection?

Worksheet for Section 2

1. Find the general solutions of the systems whose augmented matrices are given:

   (a) $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$

2. Suppose each matrix represents the augmented matrix for a system of linear equations. For each, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

   (a) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$

Worksheet for Section 3

1. Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$.

$$\mathbf{a_1} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \quad \mathbf{a_2} = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} \quad \mathbf{a_3} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$$

2. List five vectors in the Span $\{\mathbf{v_1}, \mathbf{v_2}\}$.

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

3. For what value(s) of h is $\mathbf{y}$ in the plane generated by $\mathbf{v_1}$ and $\mathbf{v_2}$?

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$$

Worksheet for Section 4

1. Compute the product using (a) the definition, and (b) the row-vector rule.

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2. Write the system as both a vector equation and a matrix equation.

$$\begin{aligned} 8x_1 - x_2 &= 4 \\ 5x_1 + 4x_2 &= 1 \\ x_1 - 3x_2 &= 2 \end{aligned}$$

3. Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A?

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

4. Describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Worksheet for Section 5

1. Determine if the system has a nontrivial solution.

$$\begin{aligned} x_1 - 3x_2 + 7x_3 &= 0 \\ -2x_1 + x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 + 9x_3 &= 0 \end{aligned}$$

2. Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where A is row equivalent to:

$$\begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$$

3. Write the solution set of the system in parametric vector form.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

4. Describe the solution set of the system in parametric vector form.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

Worksheet for Section 6

1. Find an x_1, x_2, x_3 and x_4 such that the following chemical equation is balanced.

$$(x_1)C_3H_8 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O$$

$$\mathbf{C_3H_8} = \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, \mathbf{O_2} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \mathbf{CO_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{H_2O} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

2. Determine if the columns of the matrix form a linearly independent set.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

3. Find the value(s) of h for which the vectors are linearly *dependent*.

$$\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

Worksheet for Section 7

1. Find all x in $\mathbb{R}^4$ that are mapped into the zero vector for the given matrix.

$$\begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$$

2. Let A be the matrix in the previous example. Is the vector $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \longmapsto A\mathbf{x}$?

$$\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$$

3. Let $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y_1} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\mathbf{y_2} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e_1}$ into $\mathbf{y_1}$ and $\mathbf{e_2}$ into $\mathbf{y_2}$. Find the image of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

Worksheet for Section 8

1. Find the standard matrix of T if $T : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ first reflects points through the horizontal x_1 axis and then reflects points through the line $x_2 = x_1$.

2. Let $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$.

   (a) Find the standard matrix A.

   (b) Is T one-to-one? Justify.

   (c) Is T onto? Justify.

Worksheet for Section 9

1. Compute $A - 5I_3$ and $(5I_3)A$ when:

$$\mathbf{A} = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$$

2. Let

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$ and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?

Worksheet for Section 10

1. Find the inverse of the matrix A :

   $$\mathbf{A} = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$$

2. Use the inverse found in question 1 to solve:

   $$\begin{aligned} 8x_1 + 5x_2 &= -9 \\ -7x_1 - 5x_2 &= 11 \end{aligned}$$

3. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$.

Worksheet for Section 11

1. Which of the following matrices are invertible? Justify your answers.

   (a) $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

   (b) $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$

2. An $m \times n$ **lower triangular matrix** is one whose entries *above* the main diagonal are 0's. When is a square lower triangular matrix invertible? Justify.

3. Is it possible for a 5×5 matrix to be invertible when its columns do not span $\mathbb{R}^5$? Why or why not?

4. If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.

Worksheet for Section 12

1. Find formulas for X, Y and Z in terms of A, B and C and justify your calculations. You may assume that A, X, C and Z are square.

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

2. Verify that $A^2 = I$ when $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$

3. Use partitioned matrices to show that $M^2 = I$ when

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 1 \end{bmatrix}$$

Worksheet for Section 13

1. Compute $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$ using:

   (a) cofactor expansion across the first row.

   (b) cofactor expansion down the second column.

2. Compute the determinant using cofactor expansion. Choose wisely to minimize computations.

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}$$

Worksheet for Section 14

1. Compute the determinant using the echelon form of the matrix.

$$\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}$$

2. Use determinants to find out if the matrix is invertible.

$$\begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$$

3. Use determinants to decide if the set of vectors is linearly independent.

$$\begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$$

Worksheet for Section 15

1. Let W be the union of the first and third quadrants in the xy plane. That is:

$$\mathbf{W} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \ : \ xy \geq 0 \right\}$$

   (a) If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

   (b) Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $\mathbf{u}+\mathbf{v}$ is not in W. Is W a vector space?

2. Determine if the given set is a subspace for $\mathbb{P}_n$ for the appropriate n. If not, explain why.

   (a) All polynomials of the form $\mathbf{p}(t) = at^2$, where a is in $\mathbb{R}$.

   (b) All polynomials of the form $\mathbf{p}(t) = a + t^2$, where a is in $\mathbb{R}$.

   (c) All polynomials of degree at most three with integers as coefficients.

3. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$? Why or why not?

Worksheet for Section 16

1. For the following matrix A:

$$\mathbf{A} = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$$

   (a) Find k such that Nul A is a subspace of $\mathbb{R}^k$

   (b) Find k such that Col A is a subspace of $\mathbb{R}^k$

2. With A as in exercise 1, find a nonzero vector in Nul A and a nonzero vector in Col A.

3. Let $\mathbf{A} = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Is $\mathbf{w}$ in the Col A? Is $\mathbf{w}$ in the Nul A?

Worksheet for Section 17

1. Determine which sets are a basis for $\mathbb{R}^3$. Of the sets that are not a basis, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

   (a) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$

   (b) $\begin{bmatrix}1\\-3\\0\end{bmatrix}, \begin{bmatrix}-2\\9\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-3\\5\end{bmatrix}$

2. Assume A and B are row equivalent. Find a basis for Nul A and Col A.

$$\mathbf{A} = \begin{bmatrix}-2 & 4 & -2 & -4\\ 2 & -6 & -3 & 1\\ -3 & 8 & 2 & -3\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}1 & 0 & 6 & 5\\ 0 & 2 & 5 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}$$