MATH 254 Worksheets Part 2

Worksheet for Section 1

1. Find the vector $\mathbf{x}$ determined by the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$ and the basis $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$

2. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.

   $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \; \mathbf{b}_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}, \; \mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

3. The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 3 + t - 6t^2$ relative to $\mathcal{B}$.

Worksheet for Section 2

1. Find a basis and state the dimension.

$$\left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} \ : \ a, b, c \ in \ \mathbb{R} \right\}$$

2. Find the dimensions of Nul A and Col A.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. The first four Laguerre polynomials are $1, 1-t, 2-4t+t^2$ and $6-18t+9t^2-t^3$. Show that these polynomials form a basis for $\mathbb{P}_3$.

Worksheet for Section 3

1. The matrices A and B are row equivalent. Find the following:

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) rank A

   (b) dim Nul A

   (c) basis for Col A

   (d) basis for Row A

   (e) basis for Nul A

2. Suppose a 5×6 matrix A has four pivot columns. What is the dim Nul A? Is Col $A = \mathbb{R}^4$? Why or why not?

Worksheet for Section 4

1. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue.

2. Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

3. Find a basis for the eigenspace corresponding to each eigenvalue.

   $\mathbf{A} = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$, $\lambda = 1, 5$

Worksheet for Section 5

1. Find the characteristic polynomial and the eigenvalues of $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

2. Find the characteristic polynomial of $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Worksheet for Section 6

1. Let $A = PDP^{-1}$ and compute A^4 if:

$$\mathbf{P} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

2. If the eigenvalues are $\lambda = 2, 8$ diagonalize the following matrix:

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Worksheet for Section 7

1. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \longmapsto W$ be a linear transformation with the property that:

   $$T(\mathbf{d}_1) = 2\mathbf{b}_1 - 3\mathbf{b}_2 \ , \qquad T(\mathbf{d}_2) = -4\mathbf{b}_1 + 5\mathbf{b}_2$$

   Find the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$

2. Let $T : \mathbb{P}_2 \longmapsto \mathbb{P}_3$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $(t+5)\mathbf{p}(t)$.

   (a) Find the image of $\mathbf{p}(t) = 2 - t + t^2$

   (b) Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$

Worksheet for Section 8

none

Worksheet for Section 9

1. Given that $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$ find $\mathbf{w} \bullet \mathbf{w}$, $\mathbf{x} \bullet \mathbf{w}$ and $\frac{\mathbf{x} \bullet \mathbf{w}}{\mathbf{w} \bullet \mathbf{w}}$

2. Find the distance between $\mathbf{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

3. Are $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$ orthogonal?

Worksheet for Section 10

1. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$ and express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

   $$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \text{ and } \quad \mathbf{x} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

2. Determine if $\begin{bmatrix} -.6 \\ .8 \end{bmatrix}, \begin{bmatrix} .8 \\ .6 \end{bmatrix}$ are orthonormal.

Worksheet for Section 11

1. Given that $\mathbf{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ verify that $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthogonal set and find the orthogonal projection of $\mathbf{y}$.

2. Given that $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$ and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$ Let W be the subspace spanned by the $\mathbf{u}$'s and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

Worksheet for Section 12

1. Use the Gram-Schmidt process to produce an orthogonal basis for $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$

2. Find an orthonormal basis of the subspace spanned by the vectors in number 1 above.

Worksheet for Section 13

1. Find a least squares solution of $A\mathbf{x} = \mathbf{b}$. That is, solve for $\hat{\mathbf{x}}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

Worksheet for Section 14

1. For vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ both in $\mathbb{R}^2$: let $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 5u_2v_2$.
   Show that defines an inner product.

Worksheet for Section 15

1. The matrix A has eigenvalues $\lambda = 5, 2, -2$. Orthogonally diagonalize A. That is, find the orthogonal matrix P and the diagonal matrix D.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$