1 Systems of Linear Equations

The most general form of a linear equation is:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where the a_i's are coefficients, $a_i, b \in \mathbb{R} \ \ \forall i$ and $n \in \mathbb{Z}^+$

One or more linear equations with the same variables is called a system.

ex 1

$$\begin{aligned} 2x_1 - x_2 + 4x_3 &= 12 \\ x_2 - 5x_3 &= 11 \\ x_1 + 2x_2 \qquad &= 10 \end{aligned}$$

A solution of the system makes each equation true.

To find the solution set of two linear equations graphically...

What are the options?

A system of linear equations has either:

a no solution (the system is inconsistent)

b exactly one solution (the system is consistent)

c infinitely many solutions (the system is consistent)

A linear system can be recorded compactly in a rectangular array called a *matrix*

Given the following system:

$$\begin{aligned} x_1 \qquad\quad - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

the coefficient matrix is:

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix}$$

and the augmented matrix is:

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix}^*$$

The size of a matrix is given as $m \times n$, where m represents the rows and n the columns. $*$ is a 3×4 matrix.

The question is, how do we solve the system?

i) Place the given system with an equivalent system that is easier to solve.

ii) Use the x_1 term in equation (1) to get rid of the x_1 term in equations (2) and (3).

iii) Use the x_2 term in equation (2) to get rid of the x_2 term in equations

(1) and (3).

iv) Continue this process until you run out of variables.

You will accomplish this using three basic operations:

1. You can replace a row by the sum of itself and a multiple of another row.
2. You can interchange two rows.
3. You can multiply all entries in a row by a nonzero constant.

So, lets solve:

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{2R_1 - R_2} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & -2 & -15 & 9 \\ 0 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{R_2 + 2R_3} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & -2 & -15 & 9 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Now we will start with the -5 in row 3, column 3 and move back up and to the left...

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & -2 & -15 & 9 \\ 0 & 0 & -5 & 5 \end{bmatrix} \xrightarrow[-\frac{3}{5}R_3 + R_1]{-3R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & -2 & 0 & -6 \\ 0 & 0 & -5 & 5 \end{bmatrix} \xrightarrow[-\frac{1}{5}R_3]{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \mathbf{B}$$

thus $x_1 = 5$, $x_2 = 3$ and $x_3 = -1$

A and B are *row equivalent*, that is, a sequence of row operations will transform A into B or B into A. This is useful because if the augmented matrices of two systems are row equivalent, then the two systems have **the same solution set**.

There are two fundamental questions about a linear system:

1. Is the system consistent? Does there exist a solution?
2. Is the solution unique?

You do not need to solve the system completely to determine if it is consistent. From the previous example...

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} = \ldots = \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & -2 & -15 & 9 \\ 0 & 0 & -5 & 5 \end{bmatrix}^*$$

$*$ is what is known as *triangular form*. At this point, why do we now know there is a solution?

what if:

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} = \ldots = \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & -2 & -15 & 9 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Is this system consistent? Why or why not?

Worksheet for Section 1

1. Solve the system using elementary row operations:

$$\begin{aligned} 2x_1 + 4x_2 &= -4 \\ 5x_1 + 7x_2 &= 11 \end{aligned}$$

2. Solve the system:

$$\begin{aligned} x_1 - 3x_2 \qquad\quad &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ x_2 + x_3 &= 0 \end{aligned}$$

3. Do the three planes $x_1 + 2x_2 + x_3 = 4$, $x_2 - x_3 = 1$ and $x_1 + 3x_2 = 0$ have at least one common point of intersection?

Homework for Section 1

1. Find an equation involving g, h and k that makes this augmented matrix correspond to a consistent system:

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

2. Construct three different augmented matrices for linear systems whose solution set is $x_1 = -2$, $x_2 = 1$ and $x_3 = 0$

A rectangular matrix is in row echelon form (**REF**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.

2. Each leading entry (leftmost nonzero entry) of a row is in a column to the **right** of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros.
   EXAMPLES:
   the leading entries ($\blacksquare$) may have any nonzero value
   the starred entries ($*$) may have any values

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

If a matrix in echelon form satisfies the following additional conditions, then it is in row reduced echelon form (**RREF**):

4. The leading entry in each nonzero row is 1.

5. Each leading 1 is the **only** nonzero entry in its column.
   EXAMPLES:

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

2 Row Reduction and Echelon Forms

Theorem 1

Each matrix is equivalent to one and only one reduced echelon matrix. That is, RREF's are unique.

A **pivot position** in a matrix is a location that corresponds to a leading 1 in the RREF.

A **pivot column** is a column with a pivot position.

ex 2

Given the following matrix, first obtain REF, then RREF as well as identify pivot positions and columns.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

Start with the leftmost nonzero column. So, $1_{(1,1)}$ is a pivot position. Now make zeros underneath.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \xrightarrow[-5R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}$$

The next pivot position is $-4_{(2,2)}$, get zeros underneath.

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

$-10_{(3,4)}$ is the last pivot position. Why?

To get RREF, start with the rightmost pivot position and work up and to the left creating zeros above each pivot

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix} \xrightarrow[-\frac{1}{4}R_2]{-\frac{1}{10}R_3} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-3R_3+R_2]{-7R_3+R_1} \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{and finally } \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_2+R_1} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{RREF}$$

so the original matrix is given below with the pivot positions circled and the pivot columns are 1,2 and 4.

$$\begin{bmatrix} \textcircled{1} & 3 & 5 & 7 \\ 3 & \textcircled{5} & 7 & 9 \\ 5 & 7 & 9 & \textcircled{1} \end{bmatrix}$$

let's say that an augmented matrix has the following **RREF**:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the equations are:

$x_1 - 5x_3 = 1$
$x_2 + x_3 = 4$
$0 = 0$

x_1 and x_2 correspond to pivot columns and x_3 is called a *free variable*. You can choose any value you like for x_3 since $0 = 0$ always.

So a general solution to the system is:
$x_1 = 1 + 5x_3$
$x_2 = 4 - x_3$
x_3 is free

This is called a *parametric* description of the solution since x_3 acts as a parameter.

The solution can be $(1, 4, 0)$ or $(6, 3, 1)$, etc...

If an augmented matrix has the following **RREF**:

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

What are the pivot columns? What are the basic variables? What are the free variables?

EXISTENCE AND UNIQUENESS THEOREM

A linear system is consistent if and only if the rightmost column of the augmented matrix is NOT a pivot column, that is, if and only if the **REF** has no row of the form $[0\ 0\ \ldots\ 0\ b]$ with b nonzero.

If a system is consistent it has either:
(1) a unique solution (no free variables) or
(2) infinitely many solutions (at least one free variable).

To solve a linear system, first reduce the augmented matrix to **REF** and decide if it is consistent. If so, proceed to **RREF**.

Worksheet for Section 2

1. Find the general solutions of the systems whose augmented matrices are given:

   (a) $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$

2. Suppose each matrix represents the augmented matrix for a system of linear equations. For each, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

   (a) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$

Homework for Section 2

1. Choose an h and a k such that the system has:

   (a) no solution

   (b) a unique solution

   (c) many solutions

$$x_1 + 3x_2 = 2$$

$$3x_1 + hx_2 = k$$

2. Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data (1,12),(2,15),(3,16). That is, find a_0, a_1 and a_2 such that:

$$a_0 + a_1(1) + a_2(1)^2 = 12$$

$$a_0 + a_1(2) + a_2(2)^2 = 15$$

$$a_0 + a_1(3) + a_2(3)^2 = 16$$

3 Vector Equations

A vector is simply a list of numbers. A matrix with one column is called a column vector.

ex 3

$$\begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Note that $\begin{bmatrix} 4 \\ 7 \end{bmatrix} \neq \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

The set of all vectors with two real entries is denoted $\mathbb{R}^2$

ex 4

Given that $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, find $u + v$ and $4u$

As a matter of notation, $\begin{bmatrix} 3 \\ -1 \end{bmatrix} = (3, -1)$

however $\begin{bmatrix} 3 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 3 & -1 \end{bmatrix}$

Can you give a geometric description of $\mathbb{R}^2$?

What do vectors from $\mathbb{R}^3$ look like?

Can you give a geometric description of $\mathbb{R}^3$?

Vectors from $\mathbb{R}^n$ are called ordered n-tuples and they are represented by $n \times 1$ column matrices.

ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

For all $u, v, w \in \mathbb{R}^n$ and all scalars c and d:

i. $u + v = v + u$

ii. $(u + v) + w = u + (v + w)$

iii. $u + 0 = 0 + u = u$

iv. $u + (-u) = -u + u = 0$

v. $c(u + v) = cu + cv$

vi. $(c + d)u = cu + du$

vii. $c(du) = (cd)u$

viii. $1u = u$

Given vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_p$, the vector $y = c_1v_1 + c_2v_2 + \ldots + c_pv_p$ is called a *linear combination* of $v_1, \ldots, v_p$ with weights $c_1, \ldots, c_p$

ex 5

Is $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ a linear combination of $\mathbf{a_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{a_2} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{a_3} = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$?

In other words, does there exist an x_1, x_2, x_3 such that $x_1 a_1 + x_2 a_2 + x_3 a_3 = b$?

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \iff \begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

has a solution.

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

What are the pivot columns? Is there a free variable? Is b a linear combination? Why?

Definition 1

If $v_1, \ldots, v_p$ are in $\mathbb{R}^n$, then the set of ALL linear combinations of $v_1, \ldots, v_p$ is denoted $Span\{v_1, \ldots, v_p\}$ and is called the subset of $\mathbb{R}^n$ generated or spanned by $v_1, \ldots, v_p$.

$Span\{v_1, \ldots, v_p\}$ is the collection of all vectors that can be written in the form $c_1v_1 + c_2v_2 + \ldots + c_pv_p$ with $c_1, c_2, \ldots, c_p$ scalars.

A vector b is in the $Span\{v_1, \ldots, v_p\}$ if $x_1v_1 + x_2v_2 + \ldots + x_pv_p = b$ has a solution.

Note that the zero vector must be in $Span\{v_1, \ldots, v_p\}$. Why?

ex 6

If u, v are nonzero vectors in $\mathbb{R}^3$, v not a multiple of u, what is the geometric description of $Span\{u, v\}$?

Worksheet for Section 3

1. Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$.

$$\mathbf{a_1} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \quad \mathbf{a_2} = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} \quad \mathbf{a_3} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$$

2. List five vectors in the Span $\{\mathbf{v_1}, \mathbf{v_2}\}$.

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

3. For what value(s) of h is $\mathbf{y}$ in the plane generated by $\mathbf{v_1}$ and $\mathbf{v_2}$?

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$$

Homework for Section 3

1. Construct a 3×3 matrix A, with nonzero entries, and a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $\mathbf{b}$ is *not* in the set spanned by the columns of A.

2. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and let $W = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

   (a) Is $\mathbf{b}$ in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? How many vectors are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?

   (b) Is $\mathbf{b}$ in W? How many vectors are in W?

   (c) Show that $\mathbf{a}_1$ is in W. (row operations are unnecessary here)

4 The Matrix Equation $Ax = b$

Definition 2

If A is an $m \times n$ matrix with columns $a_1, \ldots, a_n$ and if $x \in \mathbb{R}^n$,then the product Ax is a linear combination of the columns of A with x as weights.

$$Ax = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$$

Ax is defined only when the number of columns in A equals the number of entries in x

ex 7

Using the definition:

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2 \begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} + -3 \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \\ 14 \end{bmatrix} + \begin{bmatrix} -15 \\ 9 \\ -18 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

$Ax = b$ is the matrix equation

Theorem 2

A is an $m \times n$ matrix with columns $a_1, \ldots, a_n$ and $b \in \mathbb{R}^m$.

$Ax = b$ has the same solution set as the vector equation

$x_1a_1 + x_2a_2 + \ldots + x_na_n = b$

which has the same solution set as the system of linear equations with the augmented matrix

$\begin{bmatrix} a_1 & a_2 & \ldots & a_n & b \end{bmatrix}$

by definition, $Ax = b$ has a solution $\iff$ b is a linear combination of the the columns of A

ex 8

Given that $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Is the equation $Ax = b$ consistent for all possible b_1, b_2?

$$\begin{bmatrix} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & b_1 \\ 0 & 0 & b_2 + 3b_1 \end{bmatrix}$$

which means it is only consistent when $b_2 + 3b_1 = 0$, in other words, consistent for the set of all points through the origin such that $b_2 = -3b_1$

Theorem 3

If A is an $m \times n$ coefficient matrix then the following are all equivalent, that is, all true or all false.

1. for each $b \in \mathbb{R}^m$, $Ax = b$ has a solution
2. each $b \in \mathbb{R}^m$ is a linear combination of the columns of A
3. the columns of A span $\mathbb{R}^m$
4. A has a pivot position in every row

ROW VECTOR RULE TO COMPUTE Ax

If Ax is defined, the i^{th} entry in Ax is the sum of the products of the corresponding entries from row i of A and vector x

this is sometimes affectionately referred to as the "two finger rule"

ex 9

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & 2 \\ 2 & 2 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \dots$$

Theorem 4

If A is an $m \times n$ matrix with u, v vectors in $\mathbb{R}^n$ and c a scalar, then

1. $A(u+v) = Au + Av$
2. $A(cu) = c(Au)$

Worksheet for Section 4

1. Compute the product using (a) the definition, and (b) the row-vector rule.

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2. Write the system as both a vector equation and a matrix equation.

$$\begin{aligned} 8x_1 - x_2 &= 4 \\ 5x_1 + 4x_2 &= 1 \\ x_1 - 3x_2 &= 2 \end{aligned}$$

3. Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A?

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

4. Describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Homework for Section 4

1. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B below? Do the columns of B span $\mathbb{R}^3$?

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$$

2. Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when n is less than m?

3. Suppose A is a 3×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^3$ with the property that $Ax = b$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^3$.

5 Solution Sets of Linear Systems

A system is homogeneous if it can be written as $Ax = 0$ where 0 is the zero vector in $\mathbb{R}^m$

Clearly there is always at least one solution. Why?

This is called the trivial solution. The question is, does there exist a *non-trivial* solution?

That is, does there exist a nonzero vector x such that $Ax = 0$?

The homogeneous equation $Ax = 0$ has a non-trivial solution $\iff$ there is at least one free variable.

Why??

ex 10

Determine if the system has a non-trivial solution. Describe the solution set.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ -4x_1 - 9x_2 + 2x_3 &= 0 \\ - 3x_2 - 6x_3 &= 0 \end{aligned}$$

So,

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -5 & 0 \\ 0 & \textcircled{3} & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Why do we now know that a non-trivial solution exists?

$$\begin{bmatrix} \textcircled{1} & 0 & -5 & 0 \\ 0 & \textcircled{3} & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{aligned} x_1 &= 5x_3 \\ x_2 &= -2x_3 \\ x_3 &= \text{free variable} \end{aligned}$$

So the solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \quad \text{where vector } v = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

This solution is in *parametric vector form* and every solution is a scalar multiple of v. In other words, the solution set is $Span\{v\}$ and the trivial solution is when $x_3 = 0$

When a non-homogeneous system has many solutions, the general solution can be written as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

ex 11

Find the solution set in parametric vector form.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the solution is:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 + 5x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{where vector } p = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and vector } v = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

Does this look familiar? Vector v was a solution of the corresponding homogeneous equation

The solutions to $Ax = b$ are obtained by adding vector p to the solutions of $Ax = 0$. Vector p is one particular solution of $Ax = b$ corresponding to $x_3 = 0$

Theorem 5

Suppose $Ax = b$ is consistent for some b and let p be a solution. The solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$ where v_h is a solution to the homogeneous equation $Ax = 0$.

Worksheet for Section 5

1. Determine if the system has a nontrivial solution.

$$\begin{aligned} x_1 - 3x_2 + 7x_3 &= 0 \\ -2x_1 + x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 + 9x_3 &= 0 \end{aligned}$$

2. Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where A is row equivalent to:

$$\begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$$

3. Write the solution set of the system in parametric vector form.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

4. Describe the solution set of the system in parametric vector form.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

Homework for Section 5

1. Describe all solutions of $Ax = 0$ in parametric vector form, where A is row equivalent to the following:

$$\begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Prove the following:

   (a) Suppose p is a solution of $Ax = b$, so that $Ap = b$. Let v_h be any solution of the homogeneous equation $Ax = 0$ and let $w = p + v_h$. Show that w is a solution of $Ax = b$

   (b) Let w be any solution of $Ax = b$ and define $v_h = w - p$. Show that v_h is a solution of $Ax = 0$. This shows that every solution of $Ax = b$ has the form $w = p + v_h$, with p a particular solution of $Ax = b$ and v_h a solution of $Ax = 0$.

3. If A is a 3×3 matrix with two pivot positions

   (a) does the equation $Ax = 0$ have a nontrivial solution?

   (b) does the equation $Ax = b$ have at least one solution for every possible b?

4. Construct a 3x3 nonzero matrix A such that the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a solution of $Ax = 0$.

6 Linear Independence

Consider the following equation:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Do you immediately know a solution?

If so, is that the only solution?

How do you know?

Definition 3

An indexed set of vectors, $\{v_1, v_2, v_3, \ldots, v_p\}$ in $\mathbb{R}^n$ is linearly independent if $x_1v_1 + \ldots + x_pv_p = 0$ has only the trivial solution.

The set $\{v_1, v_2, v_3, \ldots, v_p\}$ is linearly dependent if there exists weights $c_1, \ldots, c_p$ not all zero such that $c_1v_1 + \ldots + c_pv_p = 0$

ex 12

Are the vectors $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$ linearly independent?

So, $\begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & -8 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -8 & 1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \implies$ no free variables

$\implies$ only the trivial solution $\implies$ linearly independent

The columns of a matrix A are linearly independent $\iff$ the equation $Ax = 0$ has ONLY the trivial solution

The set of only one vector v is linearly independent $\iff$ v is not the zero vector

A set of two vectors $\{v_1, v_2\}$ is linearly dependent if one is a multiple of the other

A set of vectors is linearly independent $\iff$ neither is a multiple of the other

What does this mean geometrically in $\mathbb{R}^2$

Theorem 6

The indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent $\iff$ at least one of the vectors is a linear combination of the others.

ex 13

$$v_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$$

1. For what values of h is v_3 in the $Span\{v_1, v_2\}$?
2. For what values of h is $\{v_1, v_2, v_3\}$ linearly dependent?

Theorem 7

If a set has more vectors then there are entries in each vector, then the set is linearly dependent.

$$\text{More variables than equations} \implies \text{there must be a free variable}$$

Theorem 8

If a set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ has the zero vector then S is linearly dependent.

Why?

Suppose $v_1 = 0$, then $1v_1 + 0v_2 + \ldots + 0v_p = 0$

ex 14

1. Which of the following sets are linearly dependent?

   (a) $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

Worksheet for Section 6

1. Find an x_1, x_2, x_3 and x_4 such that the following chemical equation is balanced.

$$(x_1)C_3H_8 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O$$

$$\mathbf{C_3H_8} = \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, \mathbf{O_2} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \mathbf{CO_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{H_2O} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

2. Determine if the columns of the matrix form a linearly independent set.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

3. Find the value(s) of h for which the vectors are linearly *dependent*.

$$\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

Homework for Section 6

1. Describe the possible echelon forms of a 2×2 matrix with linearly dependent columns.
2. Construct 3×2 matrices A and B such that $Ax = 0$ has only the trivial solution and $Bx = 0$ has a nontrivial solution.

7 Linear Transformations

Let $\mathbf{A} = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

Notice that $\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

Multiplication by the matrix A *transforms* x into b

Solving the equation $Ax = b$ is the same as finding all of the vectors x in $\mathbb{R}^4$ that are transformed into the vector b in $\mathbb{R}^2$ under the *action* of multiplication.

This is really the same concept as a function.

Definition 4

A **Transformation** (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

$\mathbb{R}^n$ is the domain, $\mathbb{R}^m$ is the co-domain and the vector $T(x)$ in $\mathbb{R}^m$ is called the *image* of x.

The set of all images is the range.

This new terminology will be useful in later applications. For now, we will focus on mappings that are associated with matrix multiplication.

ex 15

Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

If $T(x) = Ax$, find the vector x whose image under T is b. Is x unique?

$$\begin{bmatrix} Ab \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

thus $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and is unique.

ex 16

We call the following a projection transformation. Why do you think that is?

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Observe that

$$\mathbf{Ax} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

This essentially project points in $\mathbb{R}^3$ onto the (x_1, x_2) or (x, y) plane.

ex 17

We call the following a shear transformation.

$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $T = \mathbb{R}^2 \to \mathbb{R}^2$

I will show you why in class.

Definition 5

A **Transformation** T is linear if:

1. $T(u + v) = T(u) + T(v)$ $\forall$ u, v in the domain of $T(x)$
2. $T(cu) = cT(u)$ $\forall$ u and scalars c

Linear transformations preserve vector addition and scalar multiplication.

Worksheet for Section 7

1. Find all x in $\mathbb{R}^4$ that are mapped into the zero vector for the given matrix.

$$\begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$$

2. Let A be the matrix in the previous example. Is the vector $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \longmapsto A\mathbf{x}$?

$$\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$$

3. Let $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y_1} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\mathbf{y_2} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e_1}$ into $\mathbf{y_1}$ and $\mathbf{e_2}$ into $\mathbf{y_2}$. Find the image of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

Homework for Section 7

1. Suppose vectors $v_1, v_2, \ldots, v_p$ span $\mathbb{R}^n$ and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation. Suppose $T(v_i) = 0$ for $i = 1, \ldots, p$. Show that T is the zero transformation. That is, show that if x is any vector in $\mathbb{R}^n$, then $T(x) = 0$.

2. Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3 \mid x_2 \mid)$ is not linear.(provide a counterexample)

8 The Matrix of a Linear Transformation

Given a transformation T, we are usually looking for a formula. That is, find the matrix A.

The key to finding A is that T is determined by what it does to the columns of the $n \times n$ identity matrix, I_n

Theorem 9

Let $T : \mathbb{R}^n \longmapsto \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $Tx = Ax \ \forall \ x \in \mathbb{R}^n$.

ex 18

Find the standard matrix of T if $T : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ is a horizontal shear transformation that leaves e_1 unchanged and maps e_2 to $e_2 + 3e_1$

So, $T(e_1) = e_1$ and $T(e_2) = e_2 + 3e_1$

Thus, $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Definition 6

A mapping $T : \mathbb{R}^n \longmapsto \mathbb{R}^m$ is **onto** when the image of T is all of the co-domain.

That is, each vector in the co-domain is the image of at least one vector in the domain.

Definition 7

A mapping $T : \mathbb{R}^n \longmapsto \mathbb{R}^m$ is **one to one** if each vector in the co-domain, $\mathbb{R}^m$ is the image of at most one vector in the domain, $\mathbb{R}^n$.

That is, for each $b \in \mathbb{R}^n$, $T(x) = b$ has a unique solution or none at all.

Theorem 10

Let $T : \mathbb{R}^n \longmapsto \mathbb{R}^m$ be a linear transformation.

$$T \text{ is one to one} \iff T(x) = 0 \text{ has only the trivial solution.}$$

Theorem 11

Let $T : \mathbb{R}^n \longmapsto \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

$$T \text{ maps } \mathbb{R}^n \text{ onto } \mathbb{R}^m \iff \text{the columns of } A \text{ span } \mathbb{R}^m.$$

$$T \text{ is one to one} \iff \text{the columns of } A \text{ are linearly independent.}$$

ex 19

If $\mathbf{T}(\mathbf{x}) = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}$

show T is a linear transformation by finding A. Is T one to one?

Is T onto?

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three pivot positions $\Longrightarrow$ nontrivial solution $\Longrightarrow$ not one to one

No pivot position in every row $\Longrightarrow$ columns do NOT span $\mathbb{R}^4$ $\Longrightarrow$ not onto

Worksheet for Section 8

1. Find the standard matrix of T if $T : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ first reflects points through the horizontal x_1 axis and then reflects points through the line $x_2 = x_1$.

2. Let $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$.

   (a) Find the standard matrix A.

   (b) Is T one-to-one? Justify.

   (c) Is T onto? Justify.

Homework for Section 8

1. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation with A its standard matrix. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if A has how many pivot columns? Justify your answer with some theorems.

9 Matrix Operations

In terms of notation, let $A = [a_1\ a_2\ \ldots\ a_n]$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{bmatrix}$$

$a_{3,2}$ refers to the entry in the 3^{rd} row, 2^{nd} column

A *diagonal matrix* is a square matrix whose non-diagonal entries are zero. For example, I_n

Theorem 12

Let A, B and C be matrices of the same size with r and s scalars.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

Matrix multiplication is really just a composition of linear transformations.

Definition 8

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix with columns $b_1\ b_2\ \ldots\ b_p$ then

AB is $m \times p$ and $AB = A[b_1\ b_2\ \ldots\ b_p] \ = \ [Ab_1\ Ab_2\ \ldots\ Ab_p]$

The ROW COLUMN RULE says:

If AB is defined then the entry in row i, column j of AB is the sum of the products of row i of A and column j of B

ex 20

$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$

then find the entry in row 1, column 2 of AB

So,

$$\begin{bmatrix} 2 & 3 \\ * & * \end{bmatrix} \begin{bmatrix} * & 3 & * \\ * & -2 & * \end{bmatrix} \quad = \quad \begin{bmatrix} * & 0 & * \\ * & * & * \end{bmatrix}$$

Theorem 13

Let A, B and C be matrices with products defined. Then

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$ for r a scalar
5. $IA = A = AI$

Note that in general $AB \neq BA$ and also

if $AB = 0$ *that does not imply that* $A = 0$ or $B = 0$

If a matrix A is $m \times n$, the ***transpose*** of A, denoted A^T, is an $n \times m$ matrix whose columns are the corresponding rows of A.

ex 21

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\mathbf{A^T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Theorem 14

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$ for r a scalar
4. $(AB)^T = B^T A^T$

Worksheet for Section 9

1. Compute $A - 5I_3$ and $(5I_3)A$ when:

$$\mathbf{A} = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$$

2. Let

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$ and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?

Homework for Section 9

1. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA.

   Find a 3x3 matrix B, not the identity matrix or the zero matrix, such that $AB = BA$

2. Prove that $(AB)^T = B^T A^T$

10 The Inverse of a Matrix

An $n \times n$ matrix A is invertible if there exists and $n \times n$ matrix C such that $CA = I$ and $AC = I$

C is the inverse of A, denoted A^{-1}, so $A^{-1}A = I$ and $AA^{-1} = I$

A matrix that is **not** invertible is called *singular*

A matrix that is invertible is called *nonsingular*

Theorem 15

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$ then

A is invertible and $\mathbf{A^{-1}} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If $ad - bc = 0 \implies A$ is singular

$ad - bc$ is called the *determinant*, denoted $detA$

ex 22

Find A^{-1} if $\mathbf{A} = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and use it to solve

$$8x_1 + 6x_2 = 2$$
$$5x_1 + 4x_2 = -1$$

$\mathbf{A^{-1}} = \frac{1}{32-30}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$

the system above is equivalent to $Ax = b$ with $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

the solution is $x = A^{-1}b = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$

thus, $x_1 = 7$ and $x_2 = -9$

Theorem 16

1. If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1} \quad A{,}B$ invertible
3. $(A^T)^{-1} = (A^{-1})^T \quad A$ invertible

to find A^{-1}, row reduce $[AI]$ so A is row equivalent to I to get $[IA^{-1}]$

ex 23

Find A^{-1} if $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

$$[AI] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

What do you think A^{-1} is?

Worksheet for Section 10

1. Find the inverse of the matrix A :

$$\mathbf{A} = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$$

2. Use the inverse found in question 1 to solve:

$$\begin{aligned} 8x_1 + 5x_2 &= -9 \\ -7x_1 - 5x_2 &= 11 \end{aligned}$$

3. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$.

Homework for Section 10

1. Suppose A and B are nxn, B is invertible and AB is invertible. Show A is invertible by letting $C = AB$ and solving for A.

11 Characterizations of Invertible Matrices

Theorem 17

INVERTIBLE MATRIX THEOREM or **IMT**

Let A be an $n \times n$ matrix. All of the following are equivalent.

1. A is invertible
2. A is row equivalent to the $n \times n$ identity matrix
3. A has n pivot positions
4. the equation $Ax = 0$ has only the trivial solution
5. the columns of A are linearly independent
6. the linear transformation $x \longrightarrow Ax$ is one to one
7. the equation $Ax = b$ has at least one solution for each $b \in \mathbb{R}^n$
8. the columns of A span $\mathbb{R}^n$
9. the linear transformation $x \longrightarrow Ax$ is onto
10. there exists an $n \times n$ matrix C such that $CA = I$
11. there exists an $n \times n$ matrix D such that $AD = I$
12. A^T is invertible

the idea here is that we can divide all of the $n \times n$ matrices into disjoint classes

$\boxed{\text{non-singular matrices}}$ and $\boxed{\text{singular matrices}}$

each item in the IMT is a characteristic of non-singular matrices and

the negation of each item in the IMT is a characteristic of singular matrices

Theorem 18

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation and A is the standard matrix then T is invertible $\Longleftrightarrow$ A is invertible and $T^{-1} = A^{-1}x$

Worksheet for Section 11

1. Which of the following matrices are invertible? Justify your answers.

   (a) $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

   (b) $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}$

2. An $m \times n$ **lower triangular matrix** is one whose entries *above* the main diagonal are 0's. When is a square lower triangular matrix invertible? Justify.

3. Is it possible for a 5×5 matrix to be invertible when its columns do not span $\mathbb{R}^5$? Why or why not?

4. If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.

Homework for Section 11

1. Let $T(x_1, x_2) = (6x_1 - 8x_2, -5x_1 + 7x_2)$ be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that T is invertible and find a formula for T^{-1}.

12 Partitioned Matrices

We can partition a matrix into blocks

ex 24

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 & 4 & 3 & -2 \\ 1 & 1 & 3 & 9 & 7 & 1 \\ \\ -3 & 2 & 4 & 8 & 2 & 5 \end{bmatrix}$$

or

$$\mathbf{A} = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \end{bmatrix}$$

where

$$\mathbf{A_{1,1}} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix} \ , \ \mathbf{A_{2,1}} = \begin{bmatrix} -3 & 2 & 4 \end{bmatrix} \ , \ \mathbf{A_{1,2}} = \begin{bmatrix} 4 & 3 \\ 9 & 7 \end{bmatrix} \ ,$$

$$\mathbf{A_{2,2}} = \begin{bmatrix} 8 & 2 \end{bmatrix} \ , \ \mathbf{A_{1,3}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \ , \ \mathbf{A_{2,3}} = \begin{bmatrix} 5 \end{bmatrix}$$

Partitioned matrices can be multiplied using the usual row-column rules, provided it's defined of course.

ex 25

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \\ 0 & -4 & 2 & 7 & -1 \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \\ -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

thus

$$\mathbf{AB} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{1,1}B_1 + A_{1,2}B_2 \\ A_{2,1}B_1 + A_{2,2}B_2 \end{bmatrix} \cdots$$

Definition 9

A block diagonal matrix is a partitioned matrix with zero blocks off of the main diagonal of blocks.

Worksheet for Section 12

1. Find formulas for X, Y and Z in terms of A, B and C and justify your calculations. You may assume that A, X, C and Z are square.

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

2. Verify that $A^2 = I$ when $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$

3. Use partitioned matrices to show that $M^2 = I$ when

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 1 \end{bmatrix}$$

Homework for Section 12

none

13 Introduction to Determinants

First some notation...

For any square matrix A, let $A_{i,j}$ be the submatrix formed by deleting the ith row and the jth column of A.

ex 26

If

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 6 & 2 \\ 3 & -1 & 3 & -3 \\ 1 & 2 & 1 & 7 \\ 2 & 5 & 1 & 0 \end{bmatrix}$$

then

$$\mathbf{A_{3,2}} = \begin{bmatrix} 1 & 6 & 2 \\ 3 & 3 & -3 \\ 2 & 1 & 0 \end{bmatrix}$$

Definition 10

The DETERMINANT of an $n \times n$ matrix A is

$$\begin{aligned} det\ A &= a_{11}\ det\ A_{11} - a_{12}\ det\ A_{12} + \ \ldots\ (-1)^{1+n} a_{1n}\ det\ A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\ det\ A_{1j} \end{aligned}$$

ex 27

Find *det* A if

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$$

so

$$det\ A \ = \ 3\ det \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} - 0\ det \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} + 4\ det \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \ = \ \ldots \ = 1$$

Also,

$$det \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} \ = \ \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix}$$

Definition 11

The (i, j) cofactor of a matrix A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j}\ det\ A_{ij}$$

the determinant can also be computed by cofactor expansion. Using cofactors

expansion across the ith row yields

$$det\ A \ = \ a_{i1}\ C_{i1} + a_{i2}\ C_{i2} + \ \ \ldots \ \ + a_{in}\ C_{in}$$

expansion across the jth column yields

$$det\ A \ = \ a_{1j}\ C_{1j} + a_{2j}\ C_{2j} + \ \ \ldots \ \ + a_{nj}\ C_{nj}$$

the factor $(-1)^{i+j}$ gives us a checkerboard pattern.

$$\begin{bmatrix} + & - & + & - & \ldots \\ - & + & - & + & \ldots \\ + & - & + & - & \ldots \\ - & + & - & + & \ldots \\ & & . & & \\ & & . & & \\ & & . & & \end{bmatrix}$$

ex 28

Compute the following using cofactor expansion

$$\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$$

Why do you think we might want to expand across the third row?

$$det\ A \ = \ a_{31}\ C_{31} + a_{32}\ C_{32} + a_{33}\ C_{33} + a_{34}\ C_{34}$$

$$= \ 2\ (-1)^{3+1} \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix} + 0\ C_{32} + 0\ C_{33} + 0\ C_{34}$$

and now expanding that across the first row yields

$$= \ 2\ (-1)^{3+1} \left[0\ C_{11} + 0\ C_{22} + 5 \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} \right]$$

$$= \ 10$$

Theorem 19

If A is a triangular matrix, then $det\ A$ is the product of the entries on the main diagonal of A.

Worksheet for Section 13

1. Compute $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$ using:

   (a) cofactor expansion across the first row.

   (b) cofactor expansion down the second column.

2. Compute the determinant using cofactor expansion. Choose wisely to minimize computations.

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}$$

Homework for Section 13

The expansion of a 3×3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix and compute the determinant by multiplying entries on the six diagonals. Add the downward diagonal products and subtract the upward products. *This trick in no way generalizes to larger matrices.*

1.
$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$$

14 Properties of Determinants

Theorem 20

If A is a square matrix then

1. if a multiple of one row of A is added to another to produce a matrix B then
$$detB = detA$$
2. if 2 rows of A are interchanged to produce B then
$$detB = -detA$$
3. if one row of A is multiplied by k to produce B then
$$detB = k\ detA$$

ex 29

find the $detA$ if

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = 15$$

If r represents a row interchange then

$$detA = \begin{cases} (-1)^r(\text{product of the pivots}) & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible} \end{cases}$$

Theorem 21

A square matrix A is invertible $\Longleftrightarrow detA \neq 0$

Theorem 22

If A is an $n \times n$ matrix then $detA^T = detA$

Why do you think that is?

Theorem 23

If A and B are $n \times n$ matrices then $detAB = (detA)(detB)$

Worksheet for Section 14

1. Compute the determinant using the echelon form of the matrix.

$$\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}$$

2. Use determinants to find out if the matrix is invertible.

$$\begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$$

3. Use determinants to decide if the set of vectors is linearly independent.

$$\begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$$

Homework for Section 14

1. Let A and B be 4×4 matrices with $det A = -1$ and $det B = -2$. Compute:

   (a) $det\ AB$

   (b) $det\ B^5$

   (c) $det\ 2A$

   (d) $det\ A^T A$

   (e) $det\ B^{-1}AB$

2. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $det(A + B) = det\ A + det\ B \iff a + d = 0$

15 Vector Spaces and Subspaces

Definition 12

A vector space is a non-empty set V of objects, called vectors, on which are defined 2 operations, addition and multiplication by scalars (real numbers) subject to 10 axioms.

So, for all vectors u, v and w and scalars c and d

1. $u + v \in V$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. $0 \in V$ such that $u + 0 = u$
5. for each $u \in V$, there exists a $-u \in V$ such that $u + (-u) = 0$
6. $cu \in V$, c a scalar
7. $c(u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $c(du) = (cd)u$
10. $1u = u$

ex 30

Let V be the set of directed line segments in three dimensional space. Let addition be defined by the parallelogram rule and let cv be the arrow whose length is $\mid c \mid$ times the length of v in the same direction.

Show V is a vector space.

We will discuss this in class.

ex 31

For $n \geq 0$ let the set of polynomials of degree n be denoted by $\mathbb{P}_n$

Show $\mathbb{P}_n$ is a vector space.

Again, we will discuss in class.

Definition 13

A *subspace* of a vector space V is a subset H of V that has the following three properties

1. the zero vector of $V \in H$
2. H is closed under addition
3. H is closed under multiplication by scalars

Note that H is itself a vector space

ex 32

Let $\mathbb{P}$ be the set of all polynomials with real coefficients, then $\mathbb{P}_n$ is a subspace of $\mathbb{P}$.

We will formally show this in class.

ex 33

Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

ex 34

Is

$$\mathbf{H} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \quad a, b \in \mathbb{R}$$

a subspace of $\mathbb{R}^3$?

Why?

Theorem 24

If $v_1, v_2, \ldots, v_p$ are vectors from a vector space V, then $Span\{v_1, v_2, \ldots, v_p\}$ is a subspace of V

Let $v_1, v_2 \in V$ and let $\mathbf{H} = Span\{v_1, v_2\}$.

Then $0 = 0v_1 + 0v_2$ so $0 \in \mathbf{H}$

If $u = av_1 + bv_2$ and $w = cv_1 + dv_2$ then

$u + w = (a + c)v_1 + (b + d)v_2 \implies u + w \in \mathbf{H}$

also, $cu = (ca)v_1 + (cb)v_2 \implies cu \in \mathbf{H}$

Worksheet for Section 15

1. Let W be the union of the first and third quadrants in the xy plane. That is:

$$\mathbf{W} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \; : \; xy \geq 0 \right\}$$

   (a) If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

   (b) Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $\mathbf{u} + \mathbf{v}$ is not in W. Is W a vector space?

2. Determine if the given set is a subspace for $\mathbb{P}_n$ for the appropriate n. If not, explain why.

   (a) All polynomials of the form $\mathbf{p}(t) = at^2$, where a is in $\mathbb{R}$.

   (b) All polynomials of the form $\mathbf{p}(t) = a + t^2$, where a is in $\mathbb{R}$.

   (c) All polynomials of degree at most three with integers as coefficients.

3. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$. Is $\mathbf{w}$ in the subspace spanned by

   $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$? Why or why not?

Homework for Section 15

1. Let H be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Find a vector v in $\mathbb{R}^3$ such that $H = Span\{v\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

2. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

   (a) Is w in $\{v_1, v_2, v_3\}$? How many vectors are in $\{v_1, v_2, v_3\}$?

   (b) How many vectors are in the $Span\{v_1, v_2, v_3\}$?

   (c) Is w in the subspace spanned by $\{v_1, v_2, v_3\}$? Why?

3. Let $\mathbf{W} = \left\{ \begin{bmatrix} 3a + b \\ 4 \\ a - 5b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Is W a vector space? Why or why not?

16 Null Spaces and Column Spaces

Definition 14

The *null space* of an $m \times n$ matrix A, written $Nul\ A$, is the set of all solutions to $Ax = 0$

$$Nul\ A = \{x \mid x \in \mathbb{R}^n \text{ and } Ax = 0\}$$

ex 35

If

$$\mathbf{A} = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$$

is

$$\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} \in\ Nul\ A\ ?$$

All we need to do here is simply calculate Aw. Since $Aw = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ the answer is yes. That is, A "maps" w to the zero vector.

Theorem 25

The Null Space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$

This is a relatively easy proof.

Is $0 \in Nul\ A$? Of course.

If $u, v \ \in \ Nul\ A$ then $A(u+v) = Au + Av = 0 + 0 = 0$

For c a scalar, $A(cu) = c(Au) = c(0) = 0$

Thus $Nul\ A$ is a subspace of $\mathbb{R}^n$

Solving $Ax = 0$ gives us a description of $Nul\ A$

ex 36

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

so

$$[A\ 0] \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix} \implies$$

$$x_1 = 7x_3 - 6x_4$$
$$x_2 = -4x_3 + 2x_4$$
$$x_3, x_4 \text{ are free}$$

so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

and the spanning set for the $Nul\ A$ is

$$\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Definition 15

The *column space* of an $m \times n$ matrix A, written $ColA$, is the set of all linear combinations of the columns of A.

If

$$A = [a_1 \ \ldots \ a_n]$$

then

$$ColA = Span\{a_1, \ldots, a_n\}$$

Theorem 26

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$

$$ColA \ \text{ of an } mxn \text{ matrix } A \text{ is all of } \mathbb{R}^m \Longleftrightarrow$$
$$Ax = b \text{ has a solution for each } b \in \mathbb{R}^m$$

Look over the chart on the top of page 232 in the text

Definition 16

A LINEAR TRANSFORMATION T from a vector space V into a vector space W, is a rule that assigns to each vector $x \in V$ a unique vector $T(x) \in W$ such that

$$T(u+v) = T(u) + T(v) \qquad \forall\ u, v \in V$$
$$T(cu) = cT(u) \qquad \forall\ u \in V \ \text{ and scalars } c$$

The *kernel* of T is the set of all $u \in V$ such that $T(u) = 0$

If T is a matrix transformation, that is, $T(x) = Ax$ then:

$$\text{kernel } T = NulA$$
$$\text{range of } T = ColA$$

Worksheet for Section 16

1. For the following matrix A:

$$\mathbf{A} = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$$

   (a) Find k such that Nul A is a subspace of $\mathbb{R}^k$

   (b) Find k such that Col A is a subspace of $\mathbb{R}^k$

2. With A as in exercise 1, find a nonzero vector in Nul A and a nonzero vector in Col A.

3. Let $\mathbf{A} = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Is $\mathbf{w}$ in the Col A? Is $\mathbf{w}$ in the Nul A?

Homework for Section 16

1. For the following matrix A, find an explicit description of $Nul A$ by listing the vectors that span the null space.

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Find A such that the given set is $Col A$

$$\left\{ \begin{bmatrix} b - c \\ 2b + c + d \\ 5c - 4d \\ d \end{bmatrix} \; b, c, d \; \in \mathbb{R} \right\}$$

3. Let $\mathbf{A} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Is $\mathbf{w}$ in the Col A? Is $\mathbf{w}$ in the Nul A?

4. Define $T : \mathbb{P}_2 \longrightarrow \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$. For example, if $p(t) = 3 + 5t + 7t^2$ then $T(p) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$. Show that T is a linear transformation by taking arbitrary polynomials p and q in $\mathbb{P}_2$ and computing $T(p + q)$ and $T(cp)$.

17 Linear Independence and Bases

Theorem 27

An indexed set $\{v_1, v_2, \ldots, v_p\},\ v_1 \neq 0$ is linearly dependent $\iff$ some $v_j,\ j > 1$ is a linear combination of the preceding vectors $v_1, \ldots, v_{j-1}$

Definition 17

Let H be a subspace of a vector space V. An indexed set of vectors $B = \{b_1, \ldots, b_p\}$ is a basis for H if:

1. B is linearly independent
2. $H = Span\{b_1, \ldots, b_p\}$

A basis is essentially an *efficient* spanning set

ex 37

Are the following three vectors a basis for $\mathbb{R}^3$?

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$$

Since these three are equivalent to $\begin{bmatrix} 1 & 0 & 9/2 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$, what do you think?

Theorem 28

Let $S = \{v_1, \ldots, v_p\}$ and $H = Span\{v_1, \ldots, v_p\}$

1. If one of the vectors in S is a linear combination of the others, remove it. The remaining set still spans H
2. If $H \neq \{0\}$, then some subset of S is a basis for H

In other words, a basis can be constructed from a spanning set by removing unnecessary vectors.

Theorem 29

The pivot columns of a matrix A form a basis for the $Col A$

Worksheet for Section 17

1. Determine which sets are a basis for $\mathbb{R}^3$. Of the sets that are not a basis, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

   (a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$

2. Assume A and B are row equivalent. Find a basis for Nul A and Col A.

$$\mathbf{A} = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Homework for Section 17

none