ex 1 For what values of x does the following series converge?

1 Power Series

A Power Series is a series of the following form:

$$\sum_{n=0}^{\infty} c_n x^n \;=\; c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where the c_n's are the coefficients of the series and x is the variable

- for each **fixed** x we can test the series for C or D •

The sum of the series is a *function*

$$f(x) = c_0 + c_1 x + c_2 x^2 + \;\; \ldots$$

whose domain is the set of all x for which the series *converges.* Here f is a polynomial with infinitely many terms.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n \;=\; c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

is a *Power Series centered at a*

We can check for which values of x the series converges by using the

Ratio or Root Test

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \frac{n}{(x-3)^n}\right| =$$
$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \mid x-3 \mid = \mid x-3 \mid \ as \ n \longrightarrow \infty$$

Thus the series is AC when $\mid x-3 \mid < 1$ and D when $\mid x-3 \mid > 1$
So,

$$\mid x-3 \mid < 1 \implies -1 < x-3 < 1 \implies 2 < x < 4$$

Recall that the ratio and root tests give no information when the limit equals one so we MUST check the endpoints. That is what happens when $\mid x-3 \mid = 1$

Consider $x = 2$ and $x = 4$ separately:

when $x = 4 \implies \sum_{n=1}^{\infty} \frac{1}{n} \implies$ the series is a divergent harmonic

when $x = 2 \implies \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \implies$ the series converges by the alternate series test

Therefore the power series converges for $2 \leq x < 4$

Theorem

For the given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ only ONE of the following is true:

1. the series converges only when $x = a$
2. the series converges for all x
3. there exists a positive number R such that if $| x - a | < R$ the series converges and if $| x - a | > R$ the series diverges

R is called the **radius of convergence**

So in case (1) $R = 0$ and in case (2) $R = \infty$

the **interval of convergence, I,** is the interval containing all the values of x for which the series converges

from the previous example, **ex 1**, $I = [2, 4)$ and $R = 1$

ex 2 Find I and R for

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \ldots = \lim_{n\to\infty} \left| \frac{x+1}{2} \right| < 1 \iff |\,x{+}1\,| < 2 \implies R = 2$$

The interval is $(-3, 1)$ but we still need to check the endpoints

when $x = -3 \implies \sum 1 \implies$ the series diverges

when $x = 1 \implies \sum (-1)^n \implies$ the series diverges

Thus $I = (-3, 1)$

Worksheet for Section 1

1. Find the radius of convergence and interval of convergence of the series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$

Homework for Section 1

1. Solve the following:

   (a) $\int arctan\ 4t\, dt$

   (b) $\int_0^{\pi} t\ sin\ 3t\, dt$

   (c) $\int_0^{\pi/2} sin^2\ x\ cos^2\ x\, dx$

2. Find the radius of convergence, R, and the interval of convergence, I, for the following:

   (a) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

   (b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n^3}$

   (c) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

   (d) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$

   (e) $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}}$

2 Representing Functions as a Power Series

We will learn how to represent a function as a power series by manipulating a geometric series

So, recall that

$$\sum_{n=0}^{\infty} x^n \; = \; 1 + x + x^2 + x^3 + \;\; \ldots \;\; = \; \frac{1}{1-x} \quad when \;\; | \, x \, | \; < \; 1$$

ex 3 Express

$$\frac{1}{1+x^2}$$

as a power series

$$\frac{1}{1+x^2} \; = \; \frac{1}{1-(-x^2)} \; = \; \sum_{n=0}^{\infty} (-x^2)^n \; = \; \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

This geometric series converges $\iff \quad |-x^2| < 1 \quad \iff \quad x^2 < 1 \quad \iff \quad | \, x \, | < 1$
$\implies \quad I = (-1, 1)$

ex 4 Express

$$\frac{1}{x+2}$$

as a power series

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left(1-\left(-\frac{x}{2}\right)\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$$

This converges

$$\begin{aligned}
&\iff \left| -\frac{x}{2} \right| < 1 \\
&\iff |\, x \,| < 2 \\
&\implies I = (-2, 2)
\end{aligned}$$

Representations like this allow for term by term differentiation and integration

Theorem

If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$ then $f(x) = \sum c_n(x-a)^n$
is differentiable, and therefore continuous, on $(a-R, a+R)$ and

1. $f'(x) \;=\; \sum nc_n(x-a)^{n-1}$
2. $\displaystyle\int f(x)\,dx \;=\; C + \sum c_n \frac{(x-a)^{n+1}}{n+1}$

Where R is the radius of convergence for **both** (1) and (2) but I may change (you need to check)

ex 5 Find a power series representation for $ln\ (1-x)$ and the radius of convergence

Note that

$$\frac{d}{dx}\left[ln\ (1-x)\right] \;=\; \frac{-1}{1-x}$$

So

$$-ln\,(1-x) \;=\; \int \frac{1}{1-x}\,dx \;=\; \int (1+x+x^2+\,\ldots)\,dx \;=\; x+\frac{x^2}{2}+\frac{x^3}{3}+\,\ldots\,+C$$

$$=\; \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C \;\text{ therefore }\; -\,ln\;(1-x) \;=\; \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

$$\implies \; ln\,(1-x) \;=\; -\sum_{n=1}^{\infty} \frac{x^n}{n} + C \;\text{ when }\; |\,x\,| < 1 \;\; (\textit{note the change in bounds})$$

To find C let $x = 0$ and get $-ln\,1 \;=\; C \quad \implies \quad C = 0 \quad \textit{thus} \quad R = 1$

ex 6 Evaluate

$$\int \frac{1}{1+x^7}\,dx$$

Now we can do this by representing the integrand as a power series and integrating term by term. This will only be an approximation of course.

Worksheet for Section 2

1. Find a power series representation for the function and determine the interval of convergence.

   (a)
   $$f(x) = \frac{3}{1 - x^4}$$

   (b)
   $$f(x) = \frac{x}{4x + 1}$$

2. Find a power series representation for $f(x) = ln(3 + x)$

Homework for Section 2

1. Solve the following:

   (a) $\int_0^{\pi/3} tan^5\ x\ sec^4\ x\, dx$

   (b) $\int \frac{tan^3\ \theta}{cos^4\ \theta}\, d\theta$

   (c) $\int_0^1 x\sqrt{x^2+4}\, dx$

   (d) $\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\, dx$

2. Find a power series representation for the following:

   (a)
   $$f(x) = \frac{1}{1+x}$$

   (b)
   $$f(x) = \frac{x}{9+x^2}$$

3. Find a power series representation for $f(x) = ln(5-x)$ and R.

4. Evaluate $\int \frac{t}{1-t^8}\, dt$ as a power series and find R.

5. Use a power series to approximate $\int_0^{0.2} \frac{1}{1+x^5}\, dx$ to six decimal places

3 Taylor and Maclaurin Series

Recall that a power series can be expressed as

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ \ldots$$

Let's find the coefficients in terms of the function f

$$f(a) = c_0 \quad since\ all\ the\ other\ factors\ are\ 0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ \ldots \quad \Longrightarrow \quad f'(a) = c_1$$

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \ \ldots \quad \Longrightarrow \quad f''(a) = 2c_2$$

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \ \ldots \quad \Longrightarrow \quad f'''(a) = 2\cdot 3c_3 = 3!c_3$$

$$\vdots$$

In general:

$$f^{(n)}(a) = n!c_n \quad \Longrightarrow \quad c_n = \frac{f^{(n)}(a)}{n!}$$

So, if f has a power series expansion at a,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ \ldots$$

This is called the ***Taylor Series of*** f ***at*** a ***or centered at*** a

When $a = 0$, this is called a ***Maclaurin Series***

ex 7 Find the Maclaurin Series for the function $f(x) = e^x$ and the radius of convergence R

$$f(x) = e^x \quad \Longrightarrow \quad f^{(n)}(x) = e^x \;\; \forall\, x \quad \Longrightarrow \quad f^{(n)}(0) = e^0 = 1 \;\; \forall\, n$$

$$Thus \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \; = \; \sum_{n=0}^{\infty} \frac{x^n}{n!} \; = \; 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \; \ldots$$

$$R \; = \; \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \; = \; \lim_{n\to\infty} \frac{|\, x \,|}{n+1} \; = \; 0 \; < \; 1 \; always \quad \Longrightarrow \quad R = \infty$$

Note that:

A function is equal to its Taylor Series if

$$f(x) \; = \; \lim_{n\to\infty} T_n(x)$$

where $T_n(x)$ is the n^{th} degree Taylor polynomial of f at a

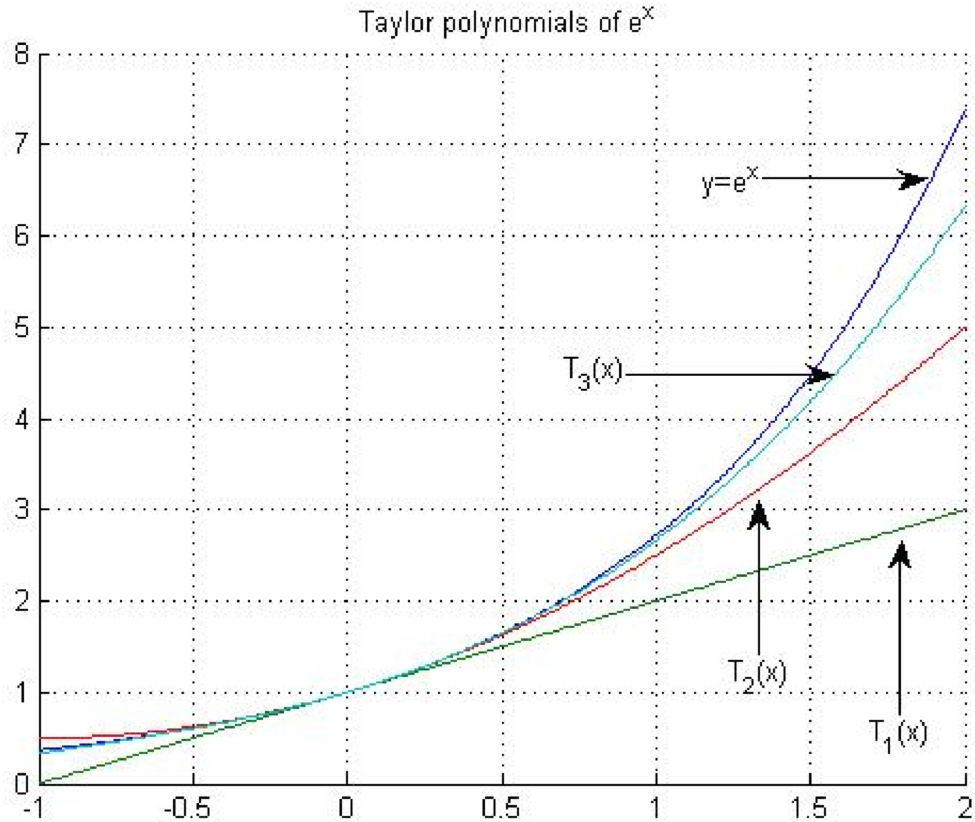
$$T_n(x) \; = \; \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i$$

$$So \quad f(x) \; \approx \; T_n(x) \qquad and \qquad T_1(x) \; = \; f(a) + f'(a)(x-a)$$

is the same as the *linearization of f*

Let $f(x) = e^x$ and let's look at $T_1(x)$, $T_2(x)$ and $T_3(x)$:

$$T_1(x) = 1 + x$$
$$T_2(x) = 1 + x + \frac{x^2}{2}$$
$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$



Note that the more terms you use the better the approximation. Why do you think that is?

Also

$$T_n(x) \;=\; \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

ex 8 Find $T_3(x)$ for $f(x) = e^x$

So

$$T_3(x) \;=\; \sum_{i=0}^{3} \frac{f^{(i)}(a)}{i!}(x-a)^i \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$that\ is \quad e^x \;\approx\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

This polynomial approximates $f(x) = e^x$ extremely well. Why? Think derivatives.

ex 9 Find the Maclaurin Series for $f(x) = cos\ x$

$$\begin{array}{rl} f(x) = cos\ x & \qquad f(0) = 1 \\ f'(x) = -sin\ x & \qquad f'(0) = 0 \\ f''(x) = -cos\ x & \qquad f''(0) = -1 \\ f'''(x) = sin\ x & \qquad f'''(0) = 0 \\ f^{(4)}(x) = cos\ x & \qquad f^{(4)}(0) = 1 \end{array}$$

So

$$cos\ x \;=\; f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \;\ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ \ldots \ = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

You will need to know the following Maclaurin Series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ \ldots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ \ldots \qquad R = \infty$$

$$sin\, x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \ldots \qquad R = \infty$$

$$cos\, x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ \ldots \qquad R = \infty$$

$$tan^{-1}\, x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ \ldots \qquad R = 1$$

ex 10 Evaluate the following integral to within .001

$$\int_0^1 e^{-x^2}\, dx$$

So

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \ \ldots \implies$$

$$\int_0^1 e^{-x^2}\, dx = \int_0^1 \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \ \ldots \right) dx$$

$$= \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \quad \ldots \right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \quad \ldots \;\approx\; .7475$$

Note that we stopped at $1/216$ since the next term was $1/1320 < .001$

Worksheet for Section 3

1. Find the Taylor series for $f(x) = ln\ x$ at $a = 2$.

2. Evaluate $\displaystyle\int \frac{sin\ x}{x}\,dx$ as an infinite series.

Homework for Section 3

1. Find the Taylor Polynomial, $T_n(x)$, for the following functions at the given value a.
   (a) $f(x) = \frac{1}{x}\ , \quad a = 2$
   (b) $f(x) = cos\ x\ , \quad a = \pi/2$
2. Approximate $f(x) = \sqrt{x}$ by a Taylor polynomial of degree 2 at $a = 4$.
3. Find the Maclaurin series as well as R for the following:
   (a) $f(x) = (1 - x)^{-2}$
   (b) $f(x) = e^{5x}$
4. Find the Taylor series representation for $f(x) = e^x$ centered at $a = 3$.
5. Evaluate $\displaystyle\int x\ cos(x^3)\, dx$ as an infinite series.