MATH 150 **Exam 6 Notes** **Spring**

1 Area and the Definite Integral

How do we find the area of a

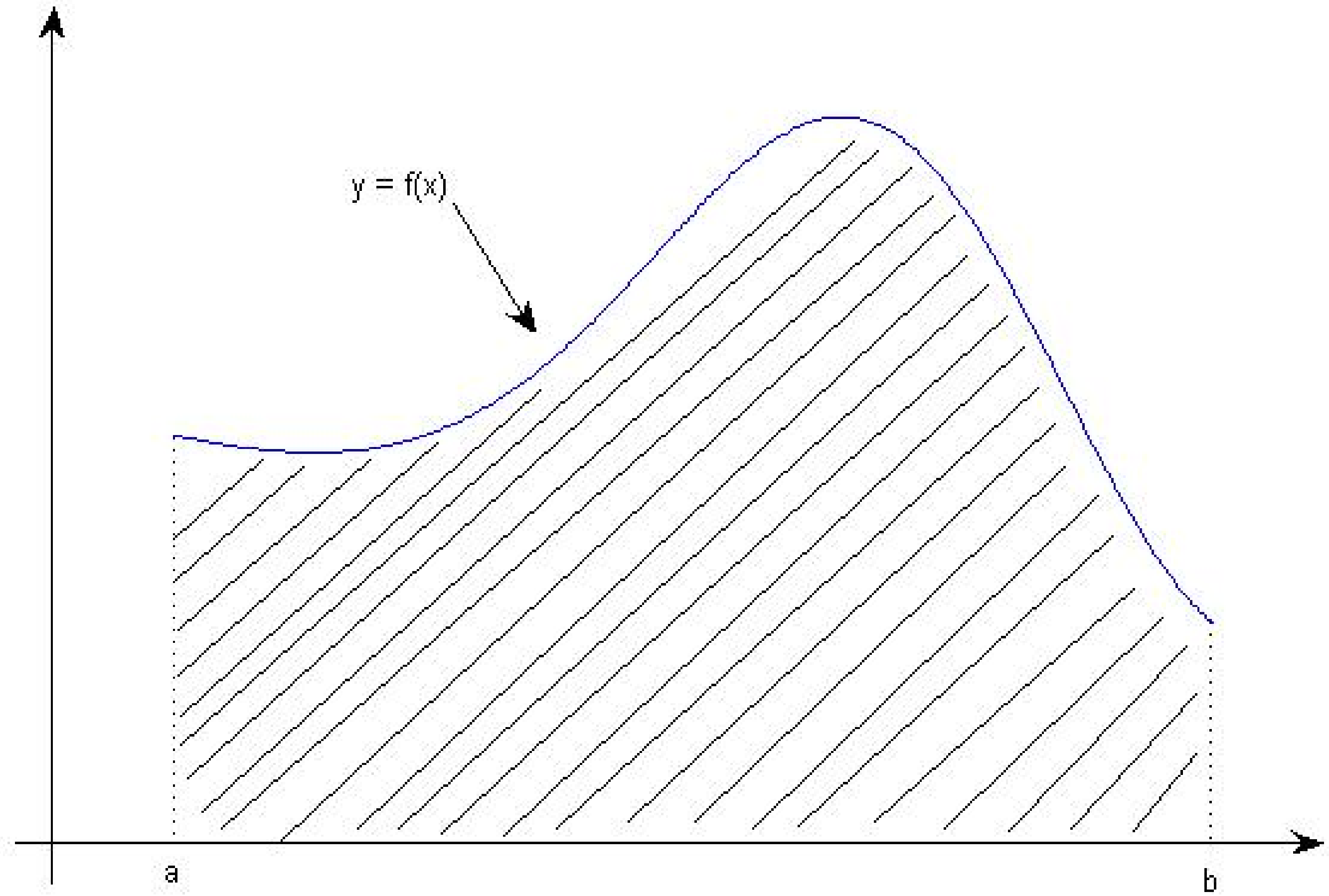
circle?

square?

rectangle?

triangle?

How about the area under a curve?



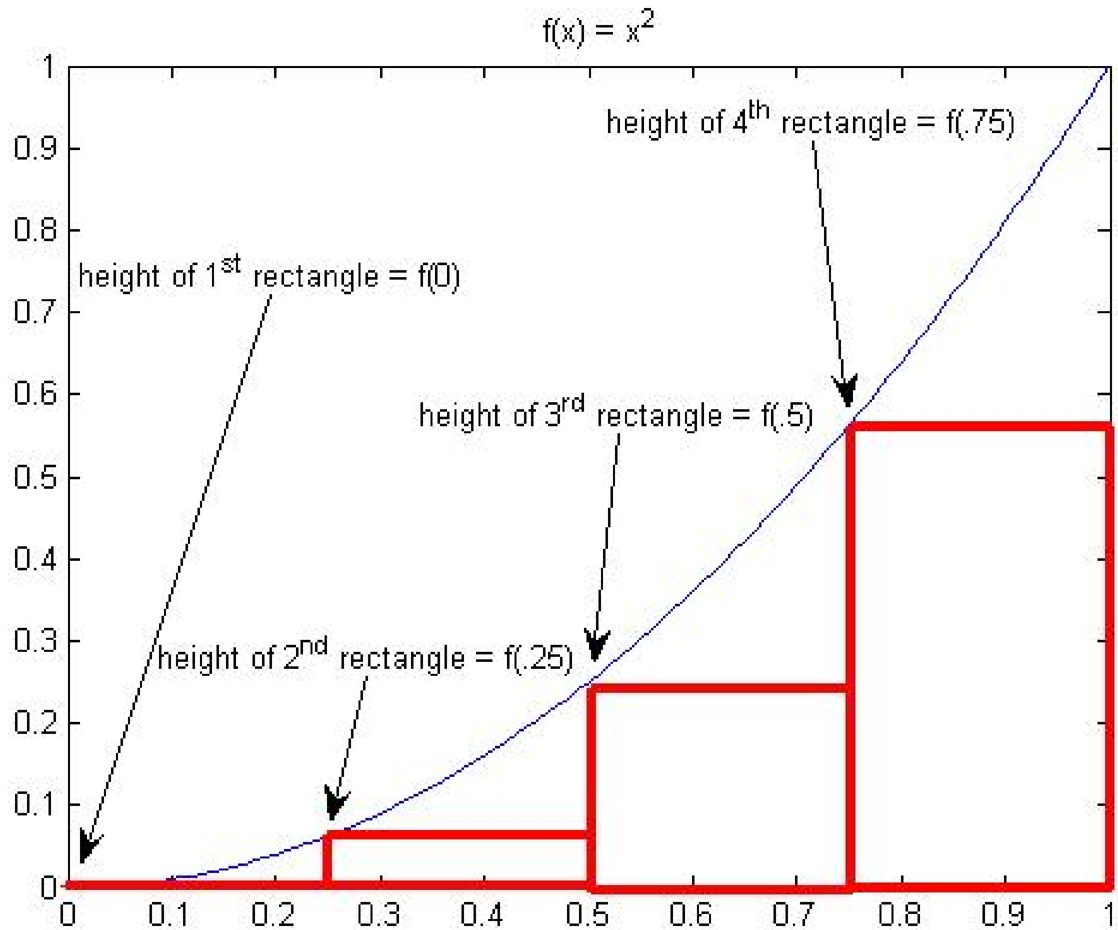
YOU BREAK THE AREA UP INTO RECTANGLES

First some notes and notation. We will be using rectangles to approximate the area under the curve. Now these rectangles need both a height AND a width. Since we will use equal width rectangles, just to make calculations easier, the widths will be determined by the question. For the heights you will use the function. I will tell you what heights to use. We will use left endpoints, denoted L_n, right endpoints, denoted R_n and midpoints, denoted M_n

Let's try some examples ...

ex 1 Let $f(x) = x^2$. Estimate the area under the curve of x^2 from $x = 0$ to $x = 1$ using four rectangles. Let's use all three heights here.

First let's use four rectangles and left endpoints for the heights



Notice that the width, denoted $\Delta\ x$, is given by $\frac{b-a}{4} = \frac{1-0}{4} = .25$. The heights of each are determined by the left endpoints. That is, since we have 4 rectangles each .25 wide, we get 4 intervals. They are $[0, .25], [.25, .5], [.5, .75]$ and $[.75, 1]$. Since we are using *left* endpoints, L_4, the first rectangle height is $f(0)$, the second rectangle has height $f(.25)$, the third $f(.5)$ and the fourth $f(.75)$.

Thus

$$L_4 = (height\,1)(width\,1) + (height\,2)(width\,2) + (height\,3)(width\,3) +$$

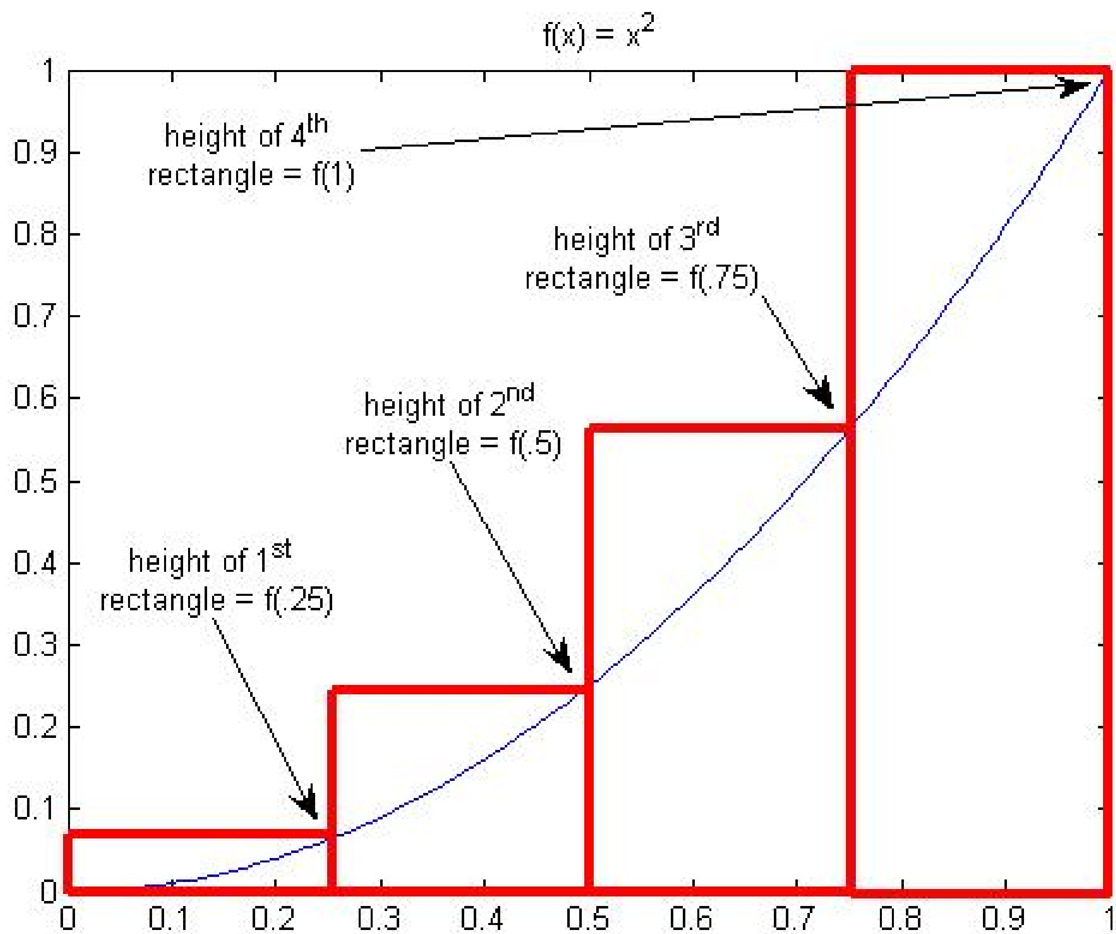
$$(height\;4)(width\;4) =$$

$$f(0)(.25) + f(.25)(.25) + f(.5)(.25) + f(.75)(.25) =$$

$$.25[(0)^2 + (.25)^2 + (.5)^2 + (.75)^2] = .21875$$

Do you think this is an overestimate or an underestimate? Why?

What if we use *right endpoints* for the heights?

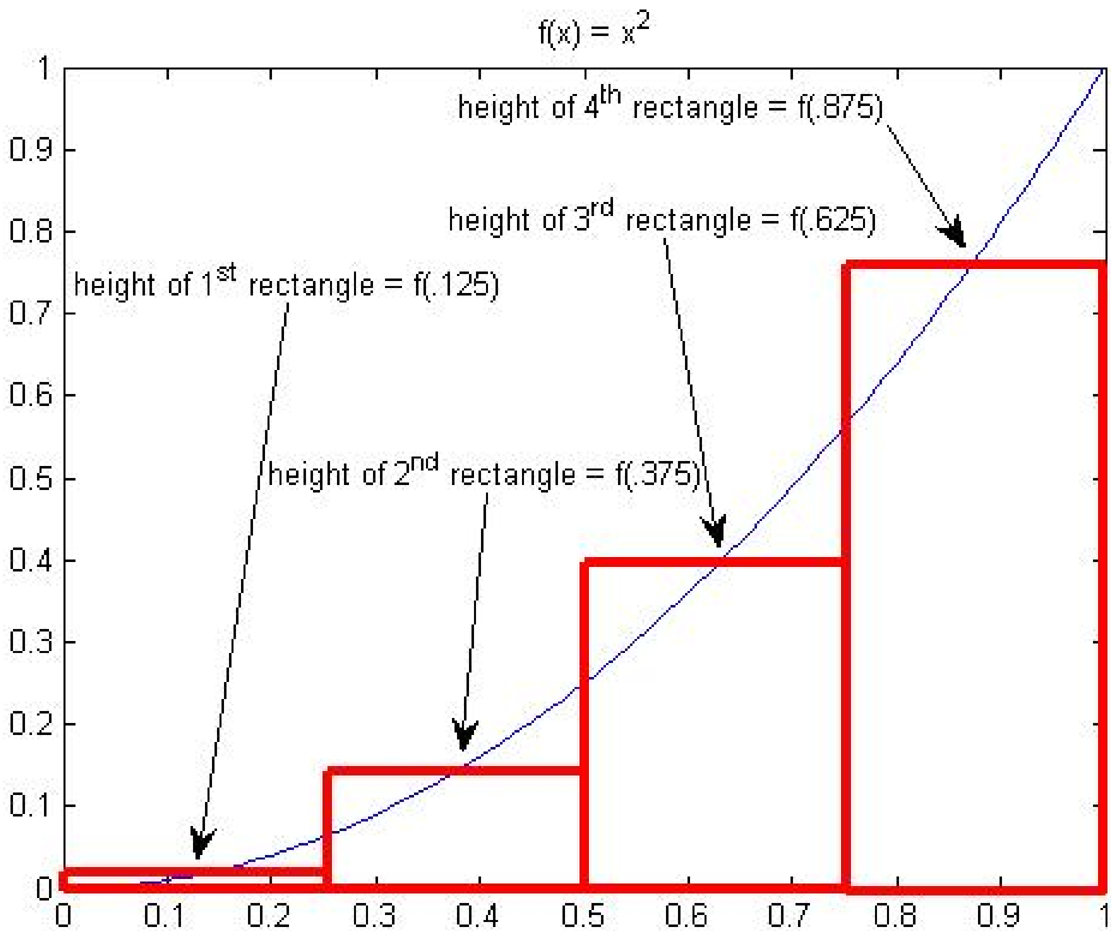


Now we have

$$R_4 = f(.25)(.25) + f(.5)(.25) + f(.75)(.25) + f(1)(.25)$$
$$= .25[(.25)^2 + (.5)^2 + (.75)^2 + (1)^2] = .46875$$

Do you think this is an overestimate or an underestimate? Why?

How about *midpoints* for the heights?

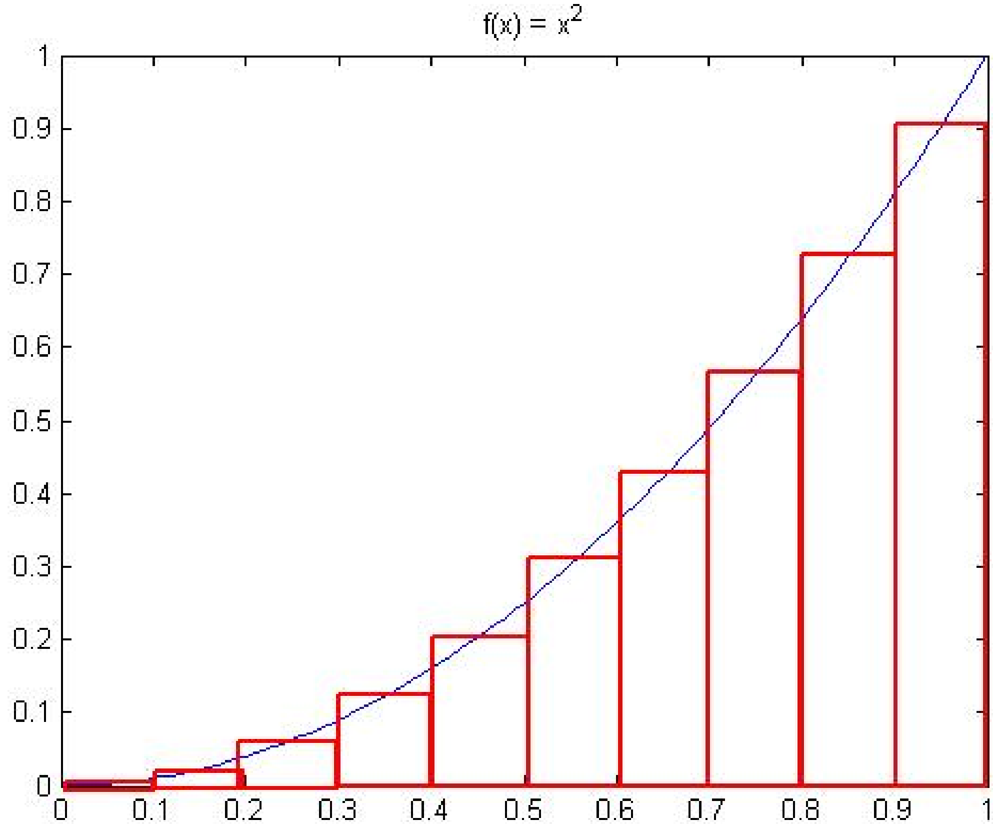


So

$$
\begin{aligned}
M_4 &= f(.125)(.25) + f(.375)(.25) + f(.625)(.25) + f(.875)(.25) \\
&= .25[(.125)^2 + (.375)^2 + (.625)^2 + (.875)^2] = .328125
\end{aligned}
$$

How do you think we can get a better estimate?

Use more rectangles. How about finding M_{10}?



Here we have that $\Delta\, x = \frac{1-0}{10} = .1$

So

$M_{10} = f(.05)(.1)+f(.15)(.1)+f(.25)(.1)+f(.35)(.1)+f(.45)(.1)+f(.55)(.1)+$

$$f(.65)(.1) + f(.75)(.1) + f(.85)(.1) + f(.95)(.1) =$$
$$.1[(.05)^2+(.15)^2+(.25)^2+(.35)^2+(.45)^2+(.55)^2+(.65)^2+(.75)^2+(.85)^2+(.95)^2]$$
$$= .3325$$

What happens if we use more rectangles?

n	M_n
4	.328125
100	.333325
200	.333331

It sure looks like the exact area is what? What are we really doing here?

How can we show that as the number of rectangles $\longrightarrow \infty$ the area gets closer to $\frac{1}{3}$?

Observe

If we have have n rectangles then the width of each rectangle is $\frac{1}{n}$

What are the heights if we use right endpoints? Since $f(x) = x^2$ and the n subintervals are:

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \ \ldots, \left[\frac{n-1}{n}, 1\right]$$

then the heights are:

$$at\ \ \frac{1}{n}\ \ the\ \ height\ \ is\ \ \left(\frac{1}{n}\right)^2$$

$$at \;\; \frac{2}{n} \;\; the \;\; height \;\; is \;\; \left(\frac{2}{n}\right)^2$$

$$at \;\; \frac{3}{n} \;\; the \;\; height \;\; is \;\; \left(\frac{3}{n}\right)^2$$

$$\vdots$$

So

$$\begin{aligned} R_n &= \frac{1}{n}\left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{n}{n}\right)^2\right) \\ &= \frac{1}{n}\cdot\frac{1}{n^2}\left(1^2 + 2^2 + \ldots + n^2\right) \end{aligned}$$

However

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

As an aside you will need to know the following:

$$\begin{aligned} 1 + 2 + 3 + \ldots + n &= \frac{n(n+1)}{2} \\ 1^2 + 2^2 + 3^2 + \ldots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^3 + 2^3 + 3^3 + \ldots + n^3 &= \left[\frac{n(n+1)}{2}\right]^2 \end{aligned}$$

Thus

$$R_n = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \quad and$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$
$$= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{3}$$

It can easily be shown that $\lim_{n\to\infty} L_n = \frac{1}{3}$ as well

So

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}$$

In general, given the area under the curve, S, divide S into n equal width strips.

The width is given by $\Delta\ x$ where $\Delta\ x = \frac{b-a}{n}$. This divides $[a, b]$ into n subintervals.

$$[x_0, x_1][x_1, x_2]\ \ldots\ [x_{n-1}, x_n] \quad where\ \ x_0 = a\ \ and\ \ x_n = b$$

The right endpoints of the subintervals are

$$x_1 = a + \Delta x\ ,\ x_2 = a + 2\Delta x\ ,\ x_3 = a + 3\Delta x \quad \ldots$$

The i^{th} rectangle has area $f(x_i)\Delta\ x$

So the area of S is approximated by the sum of the rectangles

$$A \approx R_n = f(x_1)\Delta\, x + f(x_2)\Delta\, x + f(x_3)\Delta\, x + \ldots + f(x_n)\Delta\, x$$

Definition

The area A of the region S that lies under the graph of a continuous function f is the limit of the sum of the areas of the approximating rectangles. So

$$(1) \quad A = \lim_{n\to\infty} R_n =$$
$$\lim_{n\to\infty} [f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_n)]\Delta\, x$$

or

$$(2) \quad A = \lim_{n\to\infty} L_n =$$
$$\lim_{n\to\infty} [f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_{n-1})]\Delta\, x$$

In fact you can choose any x_i^* in the i^{th} subinterval $[x_{i-1}, x_i]$

We will call the numbers x_1^* , x_2^* , $\ldots$ x_n^* SAMPLE POINTS. In general,

$$A = \lim_{n\to\infty} [f(x_1^*) + f(x_2^*) + f(x_3^*) + \ldots + f(x_n^*)]\Delta\, x$$

We will abbreviate this with *sigma notation*

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

that is

$$\sum_{i=1}^{n} f(x_i)\Delta x \ = \ f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

So the area can be written as

$$A \ = \ \lim_{n\to\infty} \ \sum_{i=1}^{n} f(x_i)\Delta x$$

or

$$A \ = \ \lim_{n\to\infty} \ \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

or

$$A \ = \ \lim_{n\to\infty} \ \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Also

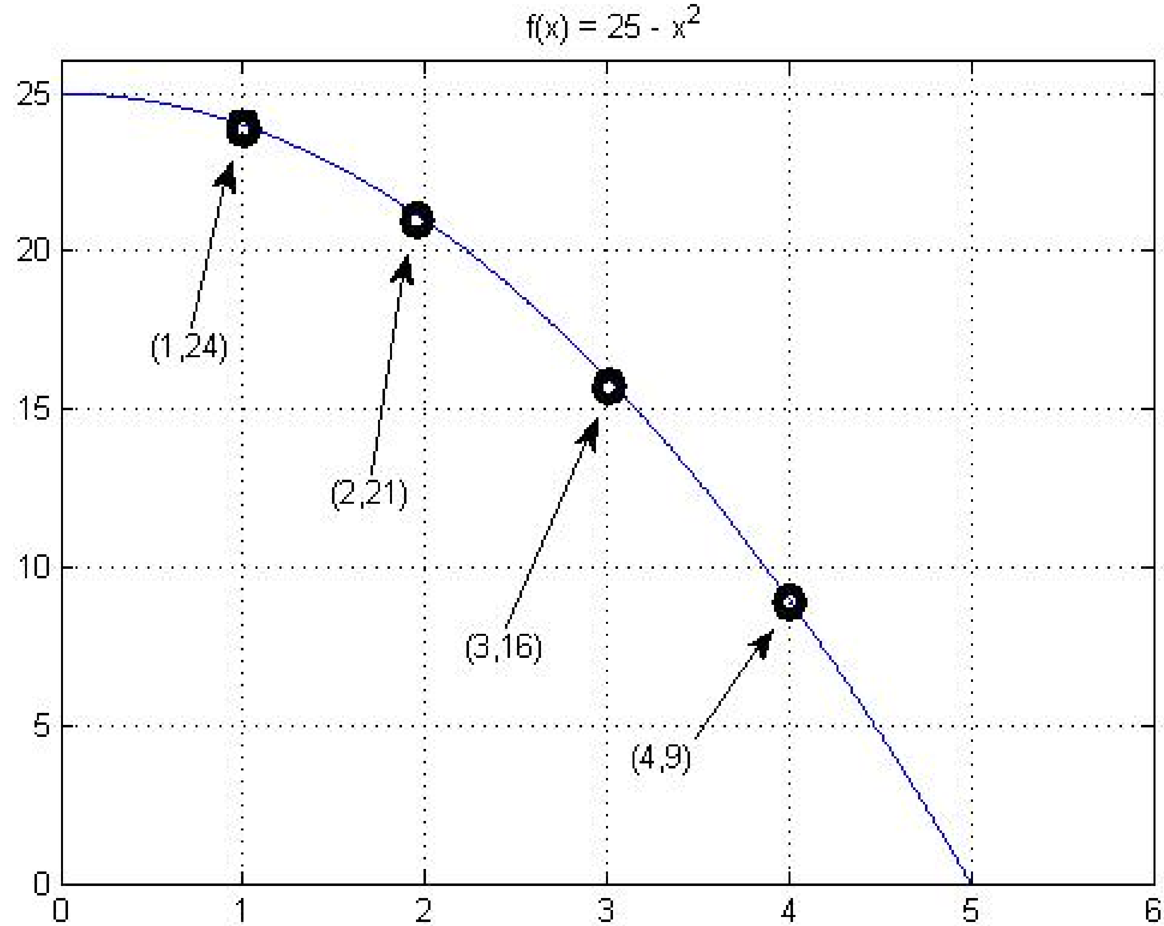
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

ex 2 For $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$

1. Estimate the area using 5 rectangles and
   (a) Right endpoints
   (b) Left endpoints
   (c) Find the area *exactly*



$$R_5 \;=\; \sum_{i=1}^{5} f(x_i)\Delta x \;=\; [f(1)+f(2)+f(3)+f(4)+f(5)]\left(\frac{5-0}{5}\right) \;=\; 70$$

$$L_5 \;=\; \sum_{i=1}^{5} f(x_{i-1})\Delta x \;=\; [f(0)+f(1)+f(2)+f(3)+f(4)]\left(\frac{5-0}{5}\right) \;=\; 95$$

To find the area exactly, observe ...

$$f(x) = 25 - x^2 \quad and \quad \Delta x \;=\; \frac{5-0}{n} \;=\; \frac{5}{n} \quad so$$

$$x_1 \;=\; \frac{5}{n}$$

$$x_2 = \frac{10}{n}$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$x_i = \frac{5i}{n}$$

Thus

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x =$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{5i}{n}\right)\frac{5}{n} =$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(25 - \left(\frac{5i}{n}\right)^2\right)\frac{5}{n} =$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(25 - \frac{25i^2}{n^2}\right)\frac{5}{n} =$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{125}{n} - \frac{125i^2}{n^3} =$$

$$\lim_{n\to\infty} \left[\frac{125}{n} \sum_{i=1}^{n} 1 - \frac{125}{n^3} \sum_{i=1}^{n} i^2\right] =$$

$$\lim_{n\to\infty} \left[\frac{125}{n}(n) - \frac{125}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right] =$$

$$125 - \frac{125}{6} \lim_{n\to\infty} 1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = 125 - \frac{125}{6}(2) = \frac{250}{3}$$

This is related to the distance problem as well. How can we find the distance traveled if the velocity is known?

ex 3

Time (s)	0	5	10	15	20	25	30
Velocity (ft/sec)	25	31	35	43	47	46	41

Can we find an estimate for the distance traveled?

$$L_6 = 5[25+31+35+43+47+46] = 1135$$
$$R_6 = 5[31+35+43+47+46+41] = 1215$$

Can we find the exact distance?

THE DISTANCE TRAVELED IS THE AREA UNDER THE CURVE OF THE VELOCITY FUNCTION

Worksheet for Section 1

1. Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using:

   (a) three rectangles and right endpoints.

   (b) six rectangles and right endpoints.

   (c) three rectangles and left endpoints.

   (d) six rectangles and left endpoints.

   (e) three rectangles and midpoints.

   (f) six rectangles and midpoints.

Homework for Section 1

1. Estimate the area under the graph of $f(x) = cos\ x$ from $x = 0$ to $x = \pi/2$ using 4 rectangles and right endpoints
2. Repeat the above problem with left endpoints
3. Oil leaked from a tank at the rate of $r(t)$ liters per hour. Find the upper and lower estimates for the total oil leaked.

t	0	2	4	6	8	10
$r(t)$	8.7	7.6	6.8	6.2	5.7	5.3

4. Write the area under the graph of $f(x) = \sqrt[4]{x}$, $1 \leq x \leq 16$ as a limit. Do NOT evaluate it.
5. Determine a region whose area is equal to the limit below. Do NOT evaluate it.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n}\ tan\left(\frac{i\pi}{4n}\right)$$

2 The Definite Integral

Definition

If f is a continuous function on $[a, b]$, divide $[a, b]$ into n subintervals of equal width, Δx, where $\Delta x = \frac{b-a}{n}$. Let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals such that x_i^* lies in the interval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

a is the lower limit, b is the upper limit and $f(x)$ is the integrand. This value represents an area, that is a number.

the sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a RIEMANN SUM

A Riemann Sum is the sum of the areas of n rectangles, it is an approximation.

The definite integral is the area under the curve, it is exact.

Also note that equal length subintervals are NOT necessary.

$$\int_a^b f(x)\,dx \quad = \quad \lim_{max\ \Delta x_i \to 0} \quad \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

ex 4 Express the following limit as an integral on $[0, \pi]$

$$\lim_{n\to\infty} \quad \sum_{i=1}^{n} (x_i^2 + x_i\ cos\ x_i)\Delta x$$

Since we will be working with sums you should know the following:

1. $\displaystyle\sum_{i=1}^{n} i \ = \ \frac{n(n+1)}{2}$

2. $\displaystyle\sum_{i=1}^{n} i^2 \ = \ \frac{n(n+1)(2n+1)}{6}$

3. $\displaystyle\sum_{i=1}^{n} i^3 \ = \ \left[\frac{n(n+1)}{2}\right]^2$

4. $\displaystyle\sum_{i=1}^{n} c \ = \ nc$

5. $\displaystyle\sum_{i=1}^{n} c\ a_i \ = \ c\sum_{i=1}^{n} a_i$

6. $\displaystyle\sum_{i=1}^{n} (a_i + b_i) \ = \ \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

7. $\displaystyle\sum_{i=1}^{n}(a_i - b_i) \;=\; \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

ex 5 Evaluate $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ by interpreting it as an area

Note that if $y = \sqrt{1-x^2} \quad\Longrightarrow\quad x^2 + y^2 = 1 \quad\Longrightarrow$ the integral represents one fourth of the unit circle.

Thus, $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx \;=\; \frac{1}{4}(\pi r^2) \;=\; \frac{\pi}{4}$

We also have the Midpoint Rule, which says $\displaystyle\int_a^b f(x)\,dx \;\approx\; \sum_{i=1}^{n} f(\overline{x_i})\Delta x$

where $\overline{x_i} \;=\; \frac{1}{2}(x_{i-1} + x_i) \;=\;$ midpoint of $[x_{i-1}, x_i]$

Some Properties

Suppose that f and g are continuous functions and that c is a constant

1. $\displaystyle\int_a^b f(x)\,dx \;=\; -\int_b^a f(x)\,dx$
2. $\displaystyle\int_a^a f(x)\,dx \;=\; 0$
3. $\displaystyle\int_a^b c\,dx \;=\; c(b-a)$
4. $\displaystyle\int_a^b cf(x)\,dx \;=\; c\int_a^b f(x)\,dx$
5. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx \;=\; \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

6. $\int_a^c f(x)\,dx \; + \; \int_c^b f(x)\,dx \; = \; \int_a^b f(x)\,dx$

7. If $f(x) \geq 0$ for $a \leq x \leq b \implies \int_a^b f(x)\,dx \; \geq \; 0$

8. If $f(x) \geq g(x)$ for $a \leq x \leq b \implies \int_a^b f(x)\,dx \; \geq \; \int_a^b g(x)\,dx$

9. If $m \leq f(x) \leq M$ for $a \leq x \leq b \implies m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

ex 6 Find $\int_6^{10} f(x)\,dx$ if $\int_0^{10} f(x)\,dx = 14$ and $\int_0^6 f(x)\,dx = 8$

Using property 6 we have that

$$\int_0^6 f(x)\,dx + \int_6^{10} f(x)\,dx = \int_0^{10} f(x)\,dx \implies 8 + \int_6^{10} f(x)\,dx = 14$$

$$\implies \int_6^{10} f(x)\,dx \; = \; 6$$

Worksheet for Section 2

1. For the following two questions, let $f(x) = x^3 - 6x$ and use endpoints $a = 0$ and $b = 3$.

   (a) Evaluate R_6

   (b) Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

Homework for Section 2

1. If $f(x) = e^x - 2\ ,\ 0 \leq x \leq 3$, find the Riemann sum with $n = 4$ correct to 6 decimal places, using midpoints.

2. Use the table below to find upper and lower estimates for $\displaystyle\int_0^{25} f(x)\,dx$

x	0	5	10	15	20	25
$f(x)$	-42	-37	-25	-6	15	36

3. Use midpoints and $n = 4$ to approximate $\displaystyle\int_2^{10} \sqrt{x^3+1}\,dx$

4. Express the following limits as definite integrals:

   (a) $\lim_{n\to\infty} \displaystyle\sum_{i=1}^{n} x_i\ ln\ (1 + x_i^2)\Delta x\ ,\quad [2, 6]$

   (b) $\lim_{n\to\infty} \displaystyle\sum_{i=1}^{n} \sqrt{2x_i^* + (x_i^*)^2}\Delta x\ ,\quad [1, 8]$

5. Calculate the following integrals **by definition**

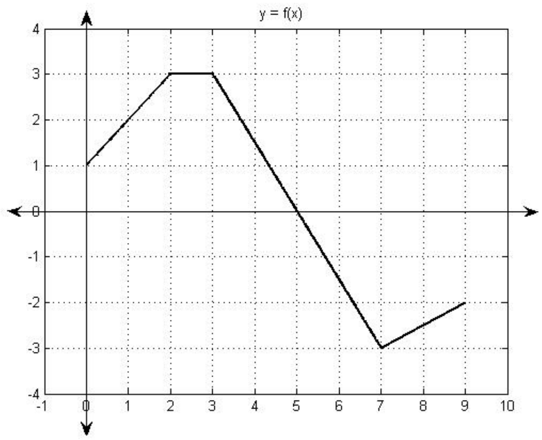
   (a) $\displaystyle\int_{-1}^{5} (1 + 3x)\,dx$

   (b) $\displaystyle\int_0^{2} (2 - x^2)\,dx$

6. Express $\displaystyle\int_2^{6} \frac{x}{1+x^5}\,dx$ as the limit of a Riemann sum. Do NOT evaluate it.

7. A graph is given below. Calculate the following by interpreting in terms of areas.

   $(a)\ \displaystyle\int_0^2 f(x)\,dx \quad (b)\ \int_0^5 f(x)\,dx \quad (c)\ \int_5^7 f(x)\,dx \quad (d)\ \int_0^9 f(x)\,dx$



8. Evaluate $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$

3 Approximate Integration

There are 2 major reasons approximation is used:

(1) Shortcuts won't work. We shall see why later ...

(2) There is no formula for the function.

Do we already know some approximation methods?

left endpoints, right endpoints, midpoints

We will now cover two other approximations that will give you much better approximations.

If you average left and right endpoints you get the:

TRAPEZOIDAL RULE

$$\int_a^b f(x)\,dx \;\approx\; \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x\right] \;=\; \frac{\Delta x}{2}\left[\sum_{i=1}^{n} f(x_{i-1}) + f(x_i)\right]$$

$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)] \;=\; T_n$$

Thus

$$\int_a^b f(x)\,dx \;\approx\; T_n \;=\; \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

The other approximation we will use, which is the most accurate, uses parabolas instead of straight line segments.

Simpson's Rule

$$\int_a^b f(x)\,dx \;\approx\; S_n \;=$$

$$\frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Where n *is even.* Why?

Worksheet for Section 3

1. Approximate

$$\int_1^2 \frac{ln\ x}{1+x}\ dx$$

   using:

   (a) M_{10}

   (b) T_{10}

   (c) S_{10}

Homework for Section 3

1. Use (i) the Trapezoidal Rule, and (ii) Simpson's Rule to approximate the following. Round to six decimal places

   (a) $\displaystyle\int_1^5 \frac{\cos\, x}{x}\, dx\ , \quad n = 8$

   (b) $\displaystyle\int_0^3 \frac{1}{1+y^5}\, dy\ , \quad n = 6$

4 The Fundamental Theorem of Calculus

Recall that:

The tangent line problem $\xrightarrow{\textit{led to}}$ DIFFERENTIATION

The area under the curve $\xrightarrow{\textit{led to}}$ INTEGRATION

Although it seems that these two are unrelated, they are actually inverses of one another!

The Fundamental Theorem of Calculus links the two and is the greatest theorem in calculus. You will memorize it.

The first part of the theorem deals with functions of the type:

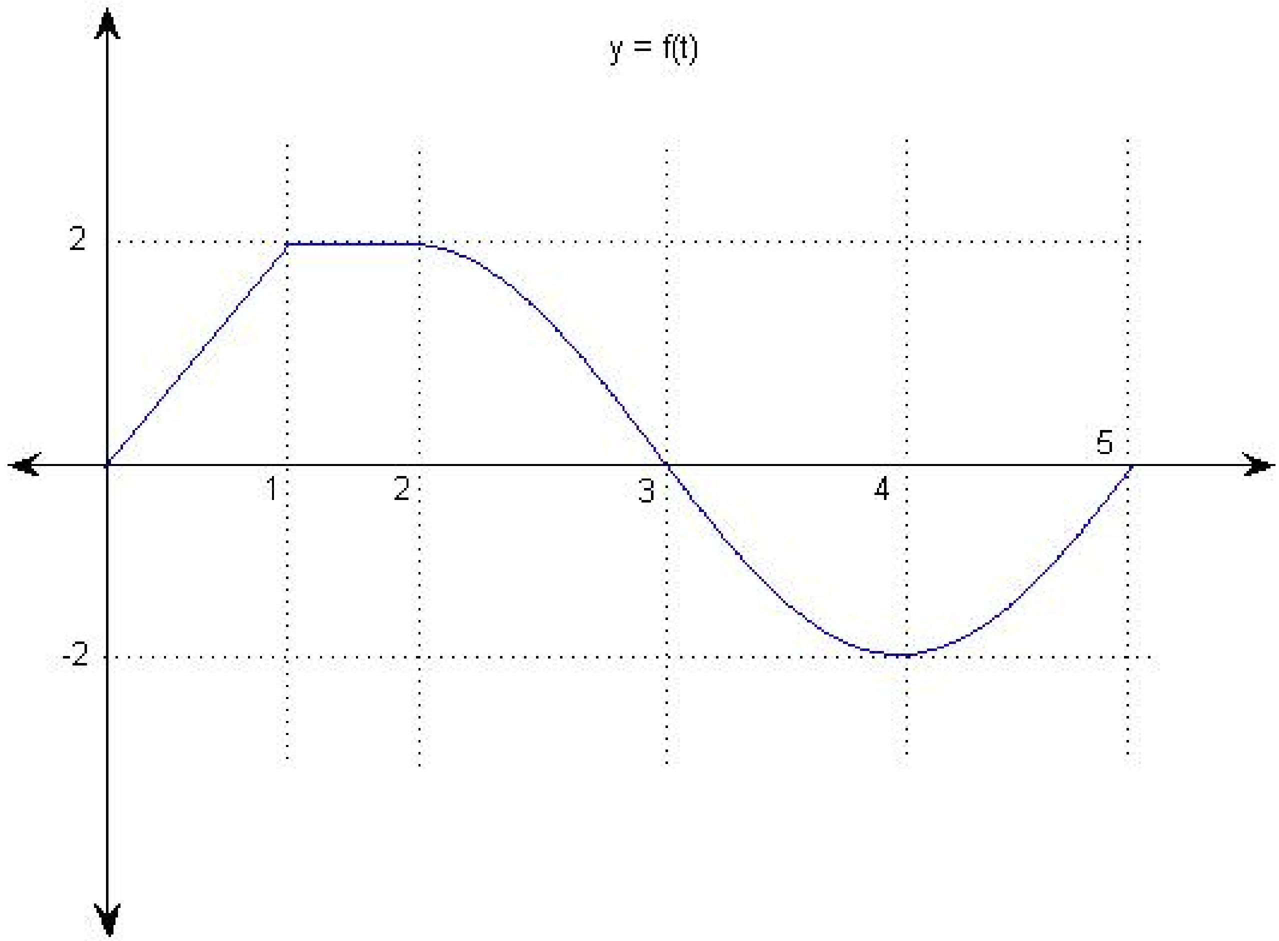
$$g(x) = \int_a^x f(t)\,dt \qquad \text{where } f \text{ is continuous on } [a,b] \text{ and } x \text{ varies from } a \text{ to } b$$

g depends <u>only</u> on x

If x is fixed, $\displaystyle\int_a^x f(t)\,dt$ is a number

Think of $g(x)$ as the "area so far"

ex 7 If $g(x) = \displaystyle\int_0^x f(t)\,dt$, find $g(1), g(2), g(3), g(4)$ and $g(5)$



$$g(x) = \int_0^x f(t)\,dt \quad \text{so}$$

$$g(1) = \int_0^1 f(t)\,dt \;=\; {}^1/_2\ \text{bh} \;=\; 1$$

$$g(2) = \int_0^2 f(t)\,dt \;=\; \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \;=\; 1+2 \;=\; 3$$

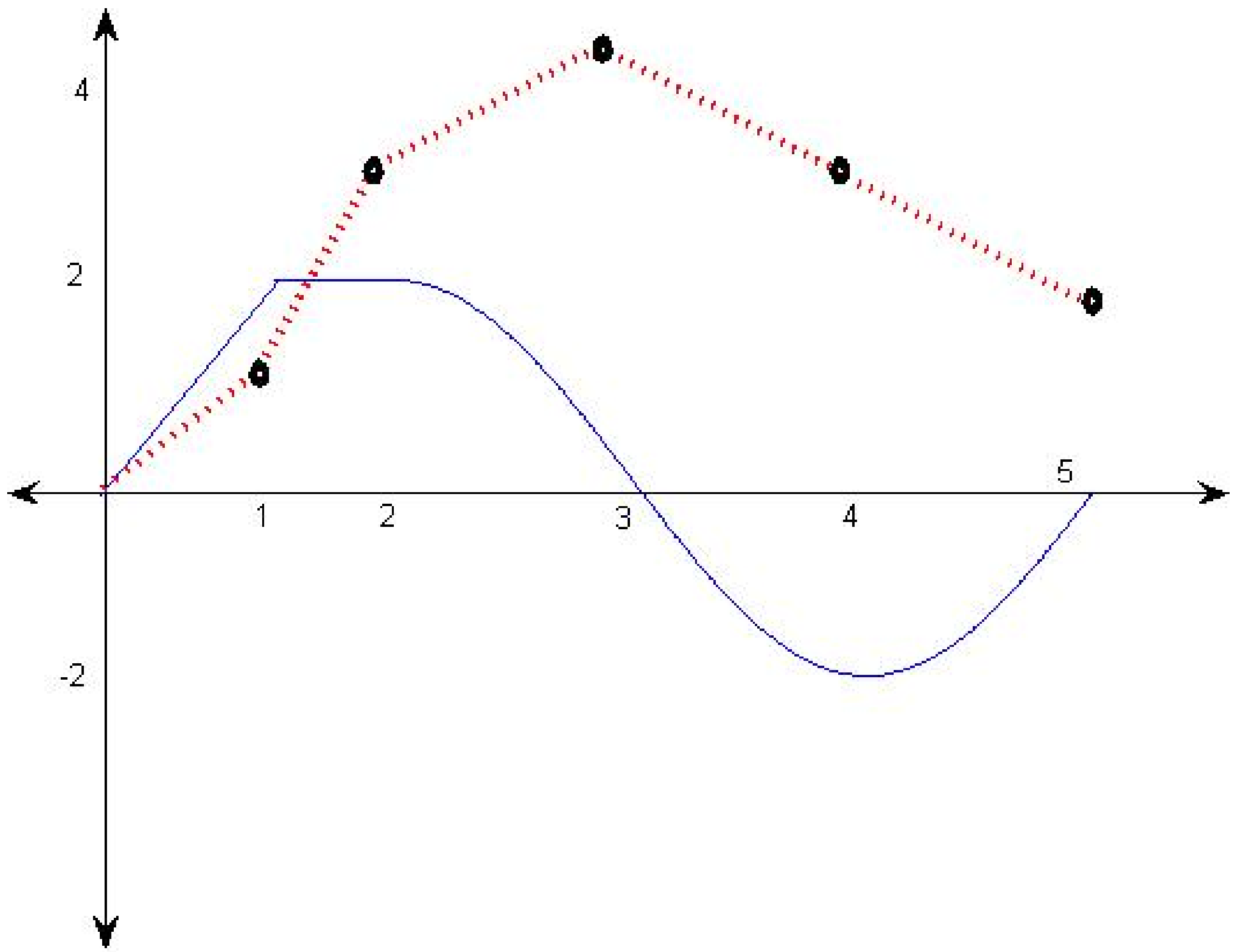
Let's estimate the area from 2 to 3 as $\approx\ 1.3$

$$g(3) = \int_0^3 f(t)\,dt \;=\; \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt \;\approx\; 3+1.3 \;=\; 4.3$$

$$g(4) = \int_0^4 f(t)\,dt \;=\; \int_0^3 f(t)\,dt + \int_3^4 f(t)\,dt \;\approx\; 4.3-1.3 \;=\; 3$$

$$g(5) = \int_0^5 f(t)\,dt \;=\; \int_0^4 f(t)\,dt + \int_4^5 f(t)\,dt \;\approx\; 3 - 1.3 \;=\; 1.7$$

If we graph both $f(t)$ and $g(x)$ on the same set of axes we would get ...



Does the graph of $g'(x)$ look familiar?

It is not a coincidence that $f = g'$!

FUNDAMENTAL THEOREM OF CALCULUS: Part I

If a function f is continuous on $[a, b]$, then the function g defined by:

$$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

Is also continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$

In other words, the derivative of a definite integral with respect to its upper limit is the integrand evaluated at its upper limit.

ex 8 Find the derivative of $g(x) = \displaystyle\int_0^x \sqrt{1+t^2}\,dt$

Since $\sqrt{1+t^2}$ is continuous, by part one of the FTC we have that

$$g'(x) = \sqrt{1+x^2}$$

Now we will work out part 2

Let

$$g(x) = \int_a^x f(t)\,dt \quad \text{and} \quad g'(x) = f(x)$$

That is, g is an antiderivative of f

Suppose that F is any other antiderivative of f on $[a, b]$. Then we know that F and g only differ by a constant. That is, $F(x) = g(x) + C$. Note that $g(a) = \displaystyle\int_a^a f(t)\,dt = 0$.

So

$$F(b)-F(a) \;=\; [g(b)+C]-[g(a)+C] \;=\; g(b)-g(a) \;=\; g(b) \;=\; \int_a^b f(t)\,dt$$

<u>FUNDAMENTAL THEOREM OF CALCULUS: Part II</u>

If a function f is continuous on $[a, b]$, then:

$$\int_a^b f(x)\,dx \;=\; F(b) - F(a) \quad \text{where} \quad F' \;=\; f$$

YOU ONLY NEED THE ENDPOINTS TO EVALUATE THE INTEGRAL!

As a matter of notation:

$$\int_a^b f(x)\,dx \;=\; F(x)\Big|_a^b \;=\; F(b) - F(a)$$

ex 9 We saw in a previous section that the area under $f(x) = x^2$ on $[0, 1]$ was $^1/_3$ Let's see ...

$$\int_0^1 x^2\;dx = \frac{x^3}{3}\Bigg|_0^1 \;=\; \frac{1^3}{3} - \frac{0^3}{3} \;=\; \frac{1}{3}$$

Why not use $F(x) = \frac{x^3}{3} + 12$ for the antiderivative? You can, try it.

ex 10 Evaluate $\displaystyle\int_1^3 e^x\ dx$

$$\int_1^3 e^x\ dx = (e^x)\Big|_1^3 = e^3 - e$$

ex 11 Find

$$\int_1^3 \frac{1}{x}\ dx$$

$$\int_1^3 \frac{1}{x}\ dx = ln\mid x\mid\Big|_1^3 = ln\mid 3\mid - ln\mid 1\mid = ln\ 3$$

ex 12 Find the area under the graph of $y = x^2$ from 0 to 2.

$$\int_0^2 x^2\ dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$$

What is the problem, if any, with the following ...

$$\int_{-1}^{3} \frac{1}{x^2}\, dx \;=\; \left.\frac{x^{-1}}{-1}\right|_{-1}^{3} \;=\; -\frac{1}{3} - 1 \;=\; -\frac{4}{3}$$

To summarize ...

<u>THE FUNDAMENTAL THEOREM OF CALCULUS:</u>

Suppose that f is continuous on $[a, b]$

1. If $g(x) = \displaystyle\int_a^x f(t)\, dt$ then $g'(x) = f(x)$

2. $\displaystyle\int_a^b f(x)\, dx \;=\; F(b) - F(a)$ where $F' \;=\; f$

INTEGRATION AND DIFFERENTIATION ARE INVERSE PROCESSES

Worksheet for Section 4

1. Use the FTC to evaluate the following integrals:

   (a) $\displaystyle\int_{-2}^{5} 6\,dx$

   (b) $\displaystyle\int_{0}^{4} (1 + 3y - y^2)\,dy$

   (c) $\displaystyle\int_{\pi}^{2\pi} cos\ \theta\,d\theta$

   (d) $\displaystyle\int_{0}^{1} \frac{4}{t^2+1}\,dt$

Homework for Section 4

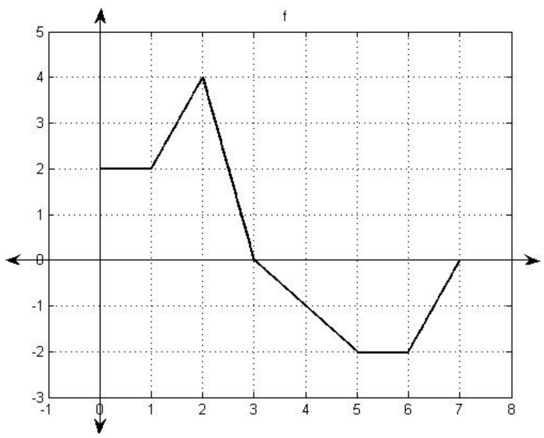
1. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is below.

    (a) Evaluate $g(0), g(1), g(2), g(3)$ and $g(6)$

    (b) On what interval is g increasing?

    (c) Where does g have a max?

    (d) Sketch g



2. Find Part I of the FTC for the following:

    (a) $g(x) = \int_1^x \frac{1}{t^3+1}\,dt$

    (b) $g(y) = \int_2^y t^2\ sin\ t\,dt$

3. Evaluate the following.

    (a) $\int_{-1}^2 (x^3 - 2x)\,dx$

    (b) $\int_1^4 (5 - 2t + 3t^2)\,dt$

    (c) $\int_0^1 x^{4/5}\,dx$

(d) $\displaystyle\int_1^2 \frac{3}{t^4}\,dt$

(e) $\displaystyle\int_0^2 x(2+x^5)\,dx$

(f) $\displaystyle\int_1^9 \frac{x-1}{\sqrt{x}}\,dx$

(g) $\displaystyle\int_1^9 \frac{1}{2x}\,dx$

(h) $\displaystyle\int_0^1 10^x\,dx$

5 Indefinite Integrals

NO LIMITS OF INTEGRATION $\longrightarrow$ indefinite integral

LIMITS OF INTEGRATION $\longrightarrow$ definite integral

$$\boxed{\int f(x)\,dx \;=\; F(x) + C \quad \text{with} \quad F'(x) = f(x)}$$

DEFINITE INTEGRAL $\longrightarrow$ *you get a number*

INDEFINITE INTEGRAL $\longrightarrow$ *you get a family of functions*

ex 13 $\displaystyle\int_0^2 x^2\,dx$

$$\int x^2\,dx \;=\; \left.\frac{x^3}{3}\right|_0^2 \;=\; \frac{8}{3}$$

ex 14 $\displaystyle\int x^2\,dx$

$$\int x^2\,dx \;=\; \frac{x^3}{3} + C$$

ex 15 $\displaystyle\int \frac{1}{x^2}\,dx$

$$\int \frac{1}{x^2}\,dx \;=\; \int x^{-2}\,dx \;=\; -\frac{1}{x} + C$$

We will adhere to the convention that $-\frac{1}{x}$ is only valid on the interval $(-\infty, 0)$ or $(0, \infty)$

TABLE OF INDEFINITE INTEGRALS:

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = ln \ \mid x \mid + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{ln\ a} + C$$

$$\int sin\ x\,dx = -\ cos\ x + C$$

$$\int cos\ x\,dx = sin\ x + C$$

$$\int sec^2\ x\,dx = tan\ x + C$$

$$\int sec\ x\ tan\ x\,dx = sec\ x + C$$

$$\int \frac{1}{1+x^2}\,dx = tan^{-1}\ x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = sin^{-1}\ x + C$$

$$\int sinh\ x\,dx = cosh\ x + C$$

$$\int cosh\ x\,dx = sinh\ x + C$$

Net Change Theorem

The integral of a rate of change is the net change. In other words

$$\int_a^b F'(x)\,dx \;=\; F(b) - F(a)$$

where F' is the rate of change and $F(b) - F(a)$ is the net change

ex 16 If the velocity of a particle is given by $v(t) = t^2 - t - 6$ in m/s, find the displacement from $1 \leq t \leq 4$ as well as the distance traveled

First we should recall that displacement and distance are slightly different. What is the difference?

$$\text{Displacement} \;=\; \int_1^4 v(t)\,dt \;=\; s(4) - s(1) \;=\; \int_1^4 t^2 - t - 6\,dt \;=$$
$$\left.\frac{t^3}{3} - \frac{t^2}{2} - 6t\right|_1^4 \;=\; \ldots \;=\; -\frac{9}{2}$$

So the particle ended up 4.5 *meters* to the *left*

$t^2 - t - 6 \;=\; (t-3)(t+2)$ so $v(t) \leq 0$ on $[1,3]$ and $v(t) \geq 0$ on $[3,4]$

So the total distance traveled is:

$$\int_1^4 \mid v(t) \mid \, dt \; = \; \int_1^3 (-v(t))\, dt \; + \; \int_3^4 v(t)\, dt$$

$$= \; \int_1^3 -t^2 + t + 6\, dt \; + \; \int_3^4 t^2 - t - 6\, dt \; = \; \frac{61}{6}$$

Worksheet for Section 5

1. Find the general indefinite integral:

   (a) $\int (x^2 + 1 + \frac{1}{x^2+1})\, dx$

   (b) $\int (3e^u + sec^2\ u)\, du$

2. Evaluate:

   (a) $\int_0^5 (2e^x + 4cos\ x)\, dx$

   (b) $\int_1^4 \sqrt{\frac{5}{x}}\, dx$

Homework for Section 5

1. Find the general indefinite integral:

   (a) $\displaystyle\int (x^2 + x^{-2})\, dx$

   (b) $\displaystyle\int (1 - t)(2 + t^2)\, dt$

   (c) $\displaystyle\int \frac{x^3 - 2\sqrt{x}}{x}\, dx$

2. Evaluate the following.

   (a) $\displaystyle\int_0^2 (6x^2 - 4x + 5)\, dx$

   (b) $\displaystyle\int_{-1}^0 (2x - e^x)\, dx$

   (c) $\displaystyle\int_{-2}^2 (3u + 1)^2\, du$

   (d) $\displaystyle\int_1^4 \sqrt{t}(1 + t)\, dt$

3. For the following velocities, find the (a) displacement and (b) distance traveled.

   (a) $v(t) = 3t - 5\ ,\quad 0 \le t \le 3$

   (b) $v(t) = t^2 - 2t - 8\ ,\quad 1 \le t \le 6$

6 Integration by Substitution

This is basically the reverse of the chain rule ... remember that?

How can we integrate

$$\int 3(3x-2)^4\,dx$$

expand?

We simplify by making a change of variable. That is,

Let $u = 3x - 2$ and that means the derivative $du = 3\,dx$

so

$$\int 3(3x-2)^4\,dx \;=\; \int (3x-2)^4\,3\;dx \;=\; \int u^4\,du$$

Which is much easier to solve.

$$\int u^4\,du \;=\; \frac{u^5}{5} + C \;=\; \frac{(3x-2)^5}{5} + C$$

This works since

$$\frac{d}{dx}\left(\frac{(3x-2)^5}{5}\right) + C \;=\; 5\;\left(\frac{(3x-2)^4}{5}\right)(3) \;=\; 3(3x-2)^4$$

What we are really doing is *decomposing* the function.

SUBSTITUTION RULE

If $u = g(x)$ is differentiable and f is continuous, then

$$\int f(g(x))g'(x)\,dx \;=\; \int f(u)\,du$$

This is how to reverse the chain rule.

ex 17 $\displaystyle\int e^{-4x}\,dx$

Let $u = -4x$ that means the derivative $du = -4\,dx$

So we need a -4 in the integral as well. There isn't one so what are we to do?

We can fix this by multiplying by -4 on the inside while multiplying by $-\frac{1}{4}$ on the outside to keep it balanced

so

$$\int e^{-4x}\,dx \;=\; \left(-\frac{1}{4}\right)\int e^{-4x}\,(-4)\,dx \;=\; \left(-\frac{1}{4}\right)\int e^{u}\,du$$

$$= \left(-\frac{1}{4}\right) e^{u} + C \;=\; \left(-\frac{1}{4}\right) e^{-4x} + C$$

ex 18 $\int e^{x^2}\, dx$

Let $u = x^2$ that means the derivative $du = 2x\, dx$

So we need a $2x$ in the integral as well. There isn't one so what are we to do?

Can we multiply inside and divide outside by a variable? **NO!**

YOU CAN ONLY MULTIPLY INSIDE AND OUTSIDE BY A CONSTANT

ex 19 $\int x^3\, sin(x^4 + 2)\, dx$

Let $u = x^4 + 2$ that means the derivative $du = 4x^3\, dx$

$$\int x^3\, sin(x^4+2)\, dx \;=\; \frac{1}{4}\int sin(x^4+2)\, 4x^3\, dx \;=$$

$$\frac{1}{4}\int sin(u)\, du$$

$$=\; -\frac{cos(u)}{4} + C \;=\; -\frac{cos(x^4+2)}{4} + C$$

ex 20 $\int tan\ x\, dx$

$$\text{Note that } \int tan\ x\, dx \ = \ \int \frac{sin\ x}{cos\ x}\, dx$$

$$= \ -\int \frac{1}{u}\, du \ = \ -\ ln \mid cos\ x \mid + C$$

ex 21 $\int x^5 \sqrt{1+x^2}\, dx$

$$\text{Let } \ u = 1 + x^2 \quad \text{then} \ \ du \ \ = 2x\ dx$$

We will need to be a little clever here, that is if

$$u = 1 + x^2 \ \implies \ x^2 = u - 1 \ \implies \ x^4 = (u-1)^2 \ \text{ thus}$$

$$\int x^5 \sqrt{1+x^2}\, dx \ = \ \frac{1}{2} \int \sqrt{u}(u-1)^2\, du \ = \ \frac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$$

$$= \ \ldots \ = \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$$

Integrals of Symmetric Functions

Suppose that f is continuous on $[-a, a]$

If f is even, $f(x) = f(-x)$, then

$$\int_{-a}^{a} f(x)\, dx \ = \ 2 \int_0^a f(x)\, dx$$

If f is odd, $-f(x) = f(-x)$, then

$$\int_{-a}^{a} f(x)\,dx \;=\; 0$$

Worksheet for Section 6

1. Find the general indefinite integral:

   (a) $\displaystyle\int 2x(x^2+3)^4\,dx$

   (b) $\displaystyle\int \frac{sin\ x}{1+cos^2\ x}\,dx$

2. Evaluate:

   (a) $\displaystyle\int_1^2 x(\sqrt{x-1})\,dx$

Homework for Section 6

1. Find the integral using the given substitution:

   (a) $\displaystyle\int x^2\sqrt{x^3+1}\,dx\ , \quad u = x^3+1$

   (b) $\displaystyle\int cos^3\,\theta\ sin\,\theta\,d\theta\ , \quad u = cos\,\theta$

2. Evaluate the indefinite integrals:

   (a) $\displaystyle\int (x+1)\sqrt{2x+x^2}\,dx$

   (b) $\displaystyle\int \frac{1}{5-3x}\,dx$

   (c) $\displaystyle\int sin(\pi t)\,dt$

   (d) $\displaystyle\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx$

   (e) $\displaystyle\int \frac{(ln\ x)^2}{x}\,dx$

   (f) $\displaystyle\int \frac{cos\sqrt{t}}{\sqrt{t}}\,dt$

   (g) $\displaystyle\int cos\,\theta\ sin^6\,\theta\,d\theta$

   (h) $\displaystyle\int e^x\sqrt{1+e^x}\,dx$

   (i) $\displaystyle\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz$

   (j) $\displaystyle\int e^{tan\ x}sec^2\ x\,dx$

   (k) $\displaystyle\int \frac{cos\ x}{sin^2\ x}\,dx$

3. Evaluate the following:

(a) $\int_0^2 (x-1)^{25}\, dx$

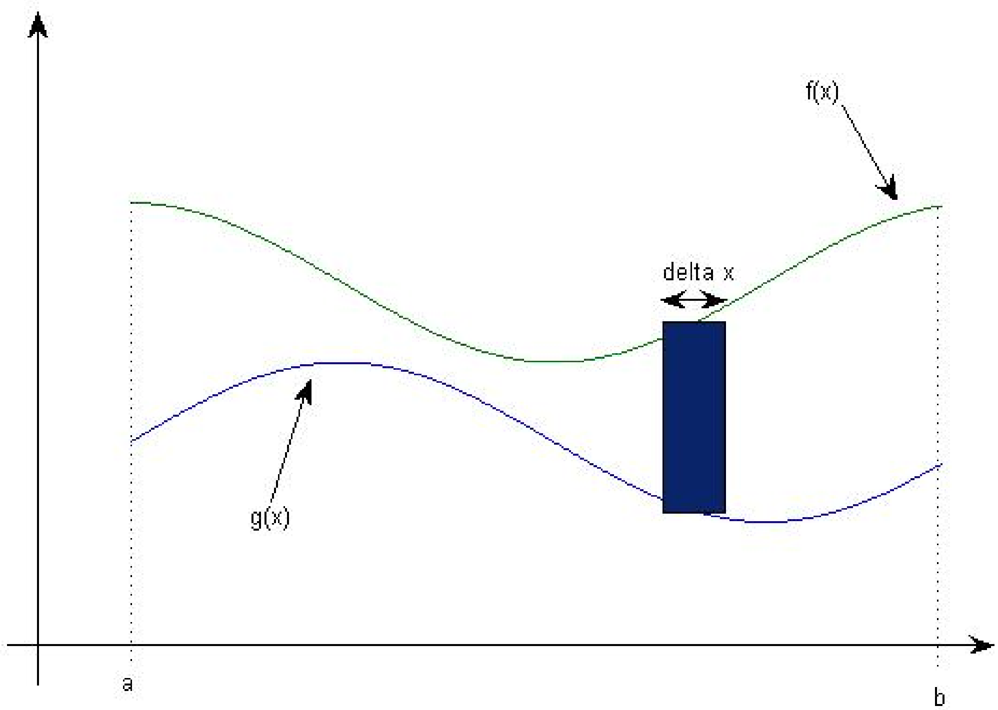
(b) $\int_0^1 x^2(1+2x^3)^5\, dx$

(c) $\int_0^{\pi} sec^2(t/4)\, dt$

(d) $\int_{-\pi/6}^{\pi/6} tan^3\ \theta\, d\theta$

(e) $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$

7 Area Between Curves



Finding the area between curves is no different from finding the area under the curve. We still use rectangles but if you examine the graph above, what is the height of that rectangle?

If $f(x) \geq g(x)$ then

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x$$

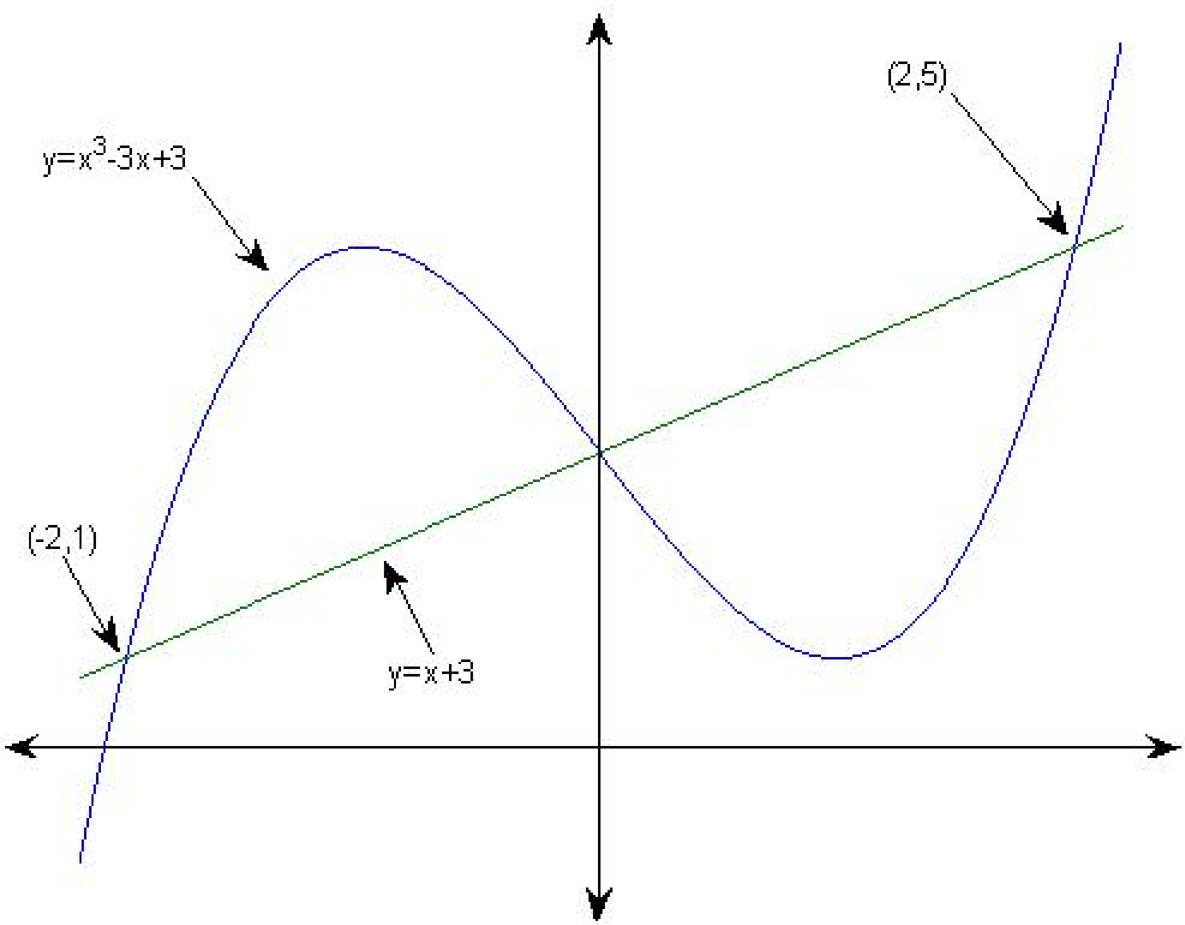
It is always the top function - the bottom function

Can the area ever be negative? Why?

The area bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ where f and g are continuous and $f \leq g \ \forall \, x \in [a, b]$ is

$$A \ = \ \int_a^b [f(x) - g(x)] \ dx$$

ex 22 Find the area between $y = x^3 - 3x + 3$ and $y = x + 3$



Remember, it is always top function - bottom function. But here we have a small problem. For one part the top function is $y = x^3 - 3x + 3$

and for the other part the top function is $y = x + 3$. The solution? Break it up into two integrals.

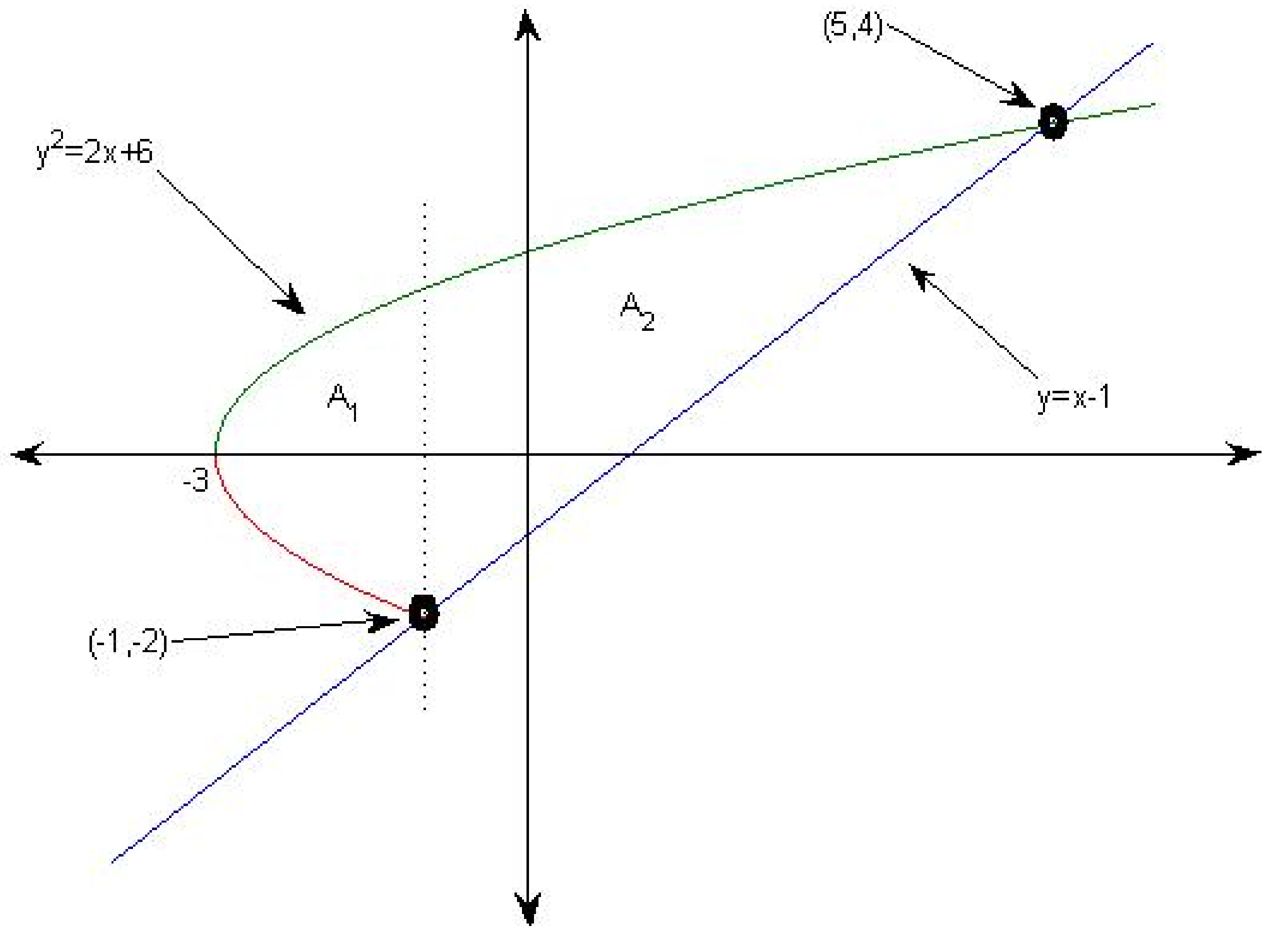
$$A = \int_{-2}^{0} (x^3 - 3x + 3) - (x + 3)\ dx + \int_{0}^{2} (x + 3) - (x^3 - 3x + 3)\ dx$$

$$= \int_{-2}^{0} (x^3 - 4x)\ dx + \int_{0}^{2} (-x^3 + 4x)\ dx$$

$$= \left(\frac{x^4}{4} - 2x^2\right)\Bigg|_{-2}^{0} + \left(-\frac{x^4}{4} + 2x^2\right)\Bigg|_{0}^{2}$$

$$= 0 - \left(\frac{(-2)^4}{4} - 2(-2)^2\right) + \left(-\frac{(2)^4}{4} + 2(2)^2\right) - 0 = -(4-8) + (-4+8) = 8$$

ex 23 Find the area between $y^2 = 2x + 6$ and $y = x - 1$



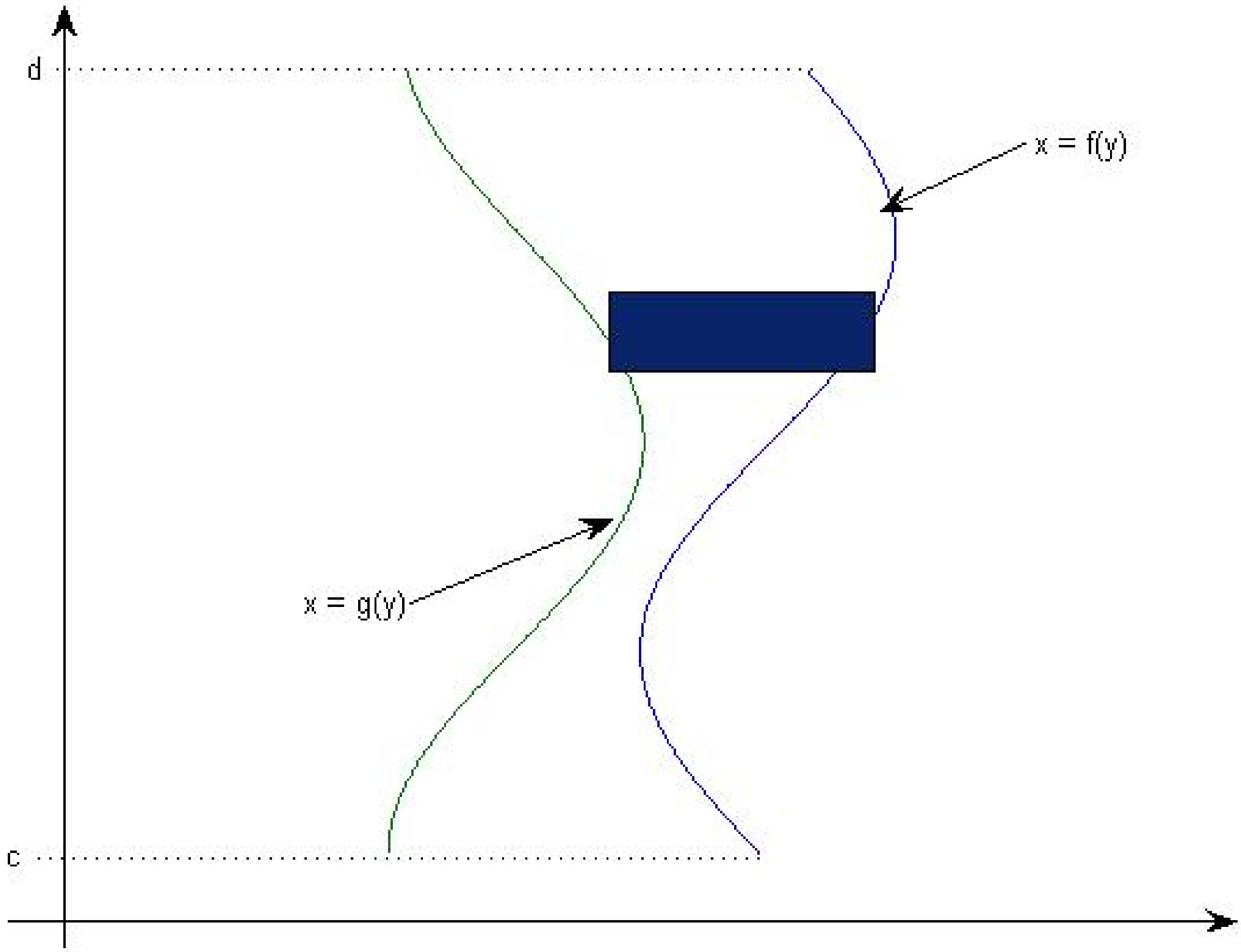
Since we need top function - bottom function we will need to break this area into 2 parts, A_1 and A_2.

$$A_1 = \int_{-3}^{-1} (\sqrt{2x+6} - \; -\sqrt{2x+6}) \; dx \quad \text{and}$$

$$A_2 = \int_{-1}^{5} (\sqrt{2x+6} - (x-1)) \; dx$$

There is a MUCH easier way ...

Regard x as a function of y



Where

$$A \;=\; \int_c^d [f(y) - g(y)]\ dy \quad \text{or} \quad A \;=\; \int_c^d [x_R - x_L]\ dy$$

from our previous example,

$$x_L \;=\; \frac{1}{2}y^2 - 3 \quad \text{and} \quad x_R \;=\; y+1 \quad \text{thus}$$

$$A \;=\; \int_{-2}^{4} [x_R - x_L]\ dy \;=\; \int_{-2}^{4} ((y+1) - (^1/_2 y^2 - 3))\ dy \;=\; \int_{-2}^{4} (-^1/_2 y^2 + y + 4)\ dy$$

$$= \; -\frac{y^3}{6} + \frac{y^2}{2} + 4y \,\Bigg|_{-2}^{4} \; = \; 18$$

Worksheet for Section 7

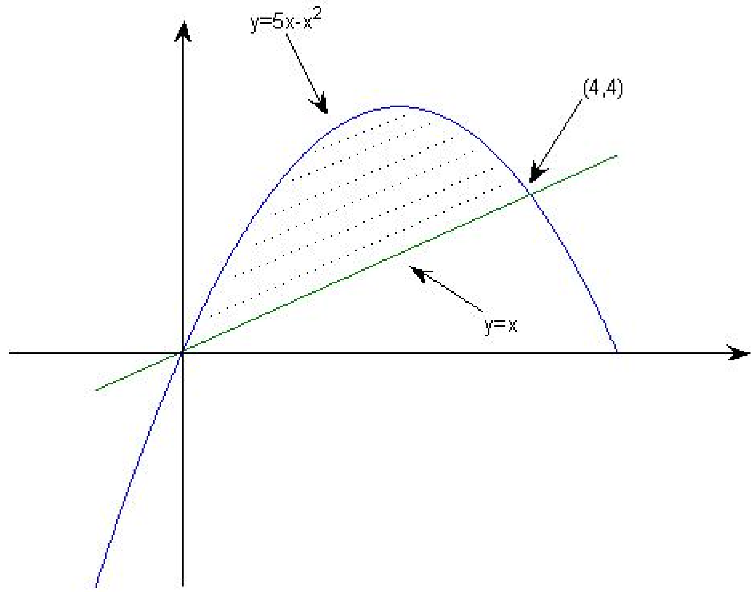
1. Sketch the region enclosed by $y = sin\ x$, $y = e^x$, $x = 0$, $x = \pi/2$. Draw a typical rectangle and find the area of the region.

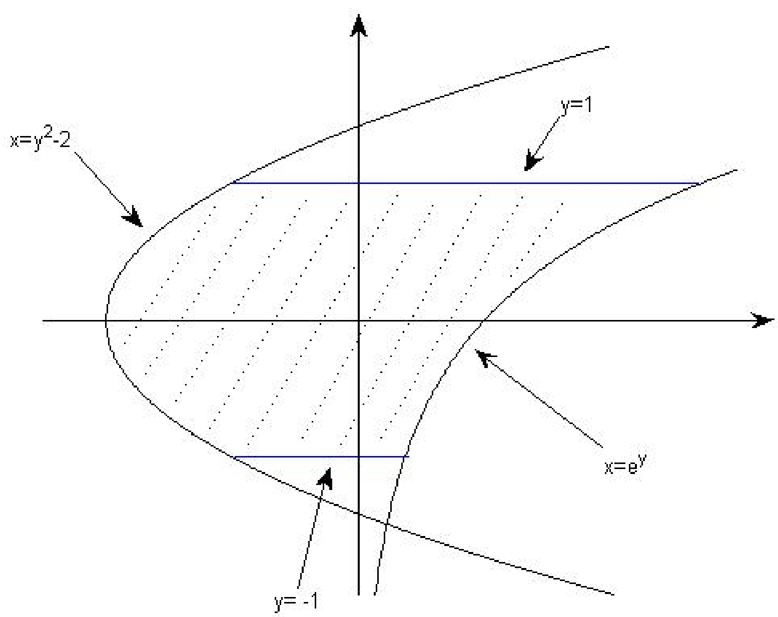
2. Sketch the region enclosed by $y = x^3 - x$, $y = 3x$. Draw a typical rectangle and find the area of the region.

3. Sketch the region enclosed by $4x + y^2 = 12$, $y = x$. Draw a typical rectangle and find the area of the region.

Homework for Section 7

1. Find the area of the following shaded regions.





2. Sketch the region, draw a typical rectangle and find the area of the following:

   (a) $y = x \ , \quad y = x^2$

   (b) $y = 1/x \ , \quad y = 1/x^2 \ , \quad x = 2$

(c) $y = 12 - x^2$, $y = x^2 - 6$

(d) $x = 1 - y^2$, $x = y^2 - 1$

(e) $y = cos\ x$, $y = sin\ 2x$, $x = 0$, $x = \pi/2$