MATH 254 Homework Part 1

Homework for Section 1

1. Find an equation involving g, h and k that makes this augmented matrix correspond to a consistent system:

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

2. Construct three different augmented matrices for linear systems whose solution set is $x_1 = -2$, $x_2 = 1$ and $x_3 = 0$

Homework for Section 2

1. Choose an h and a k such that the system has:
   (a) no solution
   (b) a unique solution
   (c) many solutions

$$x_1 + 3x_2 = 2$$
$$3x_1 + hx_2 = k$$

2. Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data (1,12),(2,15),(3,16). That is, find a_0, a_1 and a_2 such that:

$$a_0 + a_1(1) + a_2(1)^2 = 12$$
$$a_0 + a_1(2) + a_2(2)^2 = 15$$
$$a_0 + a_1(3) + a_2(3)^2 = 16$$

Homework for Section 3

1. Construct a 3x3 matrix A, with nonzero entries, and a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $\mathbf{b}$ is *not* in the set spanned by the columns of A.

2. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and let $W = Span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

   (a) Is $\mathbf{b}$ in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? How many vectors are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?

   (b) Is $\mathbf{b}$ in W? How many vectors are in W?

   (c) Show that $\mathbf{a}_1$ is in W. (row operations are unnecessary here)

Homework for Section 4

1. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B below? Do the columns of B span $\mathbb{R}^3$?

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$$

2. Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when n is less than m?

3. Suppose A is a 3x3 matrix and **b** is a vector in $\mathbb{R}^3$ with the property that $Ax = b$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^3$.

Homework for Section 5

1. Describe all solutions of $Ax = 0$ in parametric vector form, where A is row equivalent to the following:

$$\begin{bmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Prove the following:

   (a) Suppose p is a solution of $Ax = b$, so that $Ap = b$. Let v_h be any solution of the homogeneous equation $Ax = 0$ and let $w = p + v_h$. Show that w is a solution of $Ax = b$

   (b) Let w be any solution of $Ax = b$ and define $v_h = w - p$. Show that v_h is a solution of $Ax = 0$. This shows that every solution of $Ax = b$ has the form $w = p + v_h$, with p a particular solution of $Ax = b$ and v_h a solution of $Ax = 0$.

3. If A is a 3x3 matrix with two pivot positions

   (a) does the equation $Ax = 0$ have a nontrivial solution?

   (b) does the equation $Ax = b$ have at least one solution for every possible b?

4. Construct a 3x3 nonzero matrix A such that the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a solution of $Ax = 0$.

Homework for Section 6

1. Describe the possible echelon forms of a 2x2 matrix with linearly dependent columns.

2. Construct 3x2 matrices A and B such that $Ax = 0$ has only the trivial solution and $Bx = 0$ has a nontrivial solution.

Homework for Section 7

1. Suppose vectors $v_1, v_2, \ldots, v_p$ span $\mathbb{R}^n$ and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation. Suppose $T(v_i) = 0$ for $i = 1, \ldots, p$. Show that T is the zero transformation. That is, show that if x is any vector in $\mathbb{R}^n$, then $T(x) = 0$.

2. Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3 \mid x_2 \mid)$ is not linear.(provide a counterexample)

Homework for Section 8

1. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation with A its standard matrix. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if A has how many pivot columns? Justify your answer with some theorems.

Homework for Section 9

1. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA. Find a 3x3 matrix B, not the identity matrix or the zero matrix, such that $AB = BA$

2. Prove that $(AB)^T = B^T A^T$

Homework for Section 10

1. Suppose A and B are nxn, B is invertible and AB is invertible. Show A is invertible by letting $C = AB$ and solving for A.

Homework for Section 11

1. Let $T(x_1, x_2) = (6x_1 - 8x_2, -5x_1 + 7x_2)$ be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. Show that T is invertible and find a formula for T^{-1}.

Homework for Section 12

none

Homework for Section 13

The expansion of a 3x3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix and compute the determinant by multiplying entries on the six diagonals. Add the downward diagonal products and subtract the upward products. *This trick in no way generalizes to larger matrices.*

1.
$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$$

Homework for Section 14

1. Let A and B be 4x4 matrices with $det A = -1$ and $det B = -2$. Compute:

   (a) $det\ AB$

   (b) $det\ B^5$

   (c) $det\ 2A$

   (d) $det\ A^T A$

   (e) $det\ B^{-1}AB$

2. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $det(A+B) = det\ A + det\ B \iff a+d = 0$

Homework for Section 15

1. Let H be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Find a vector v in $\mathbb{R}^3$ such that $H = Span\{v\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

2. Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

   (a) Is w in $\{v_1, v_2, v_3\}$? How many vectors are in $\{v_1, v_2, v_3\}$?

   (b) How many vectors are in the $Span\{v_1, v_2, v_3\}$?

   (c) Is w in the subspace spanned by $\{v_1, v_2, v_3\}$? Why?

3. Let $\mathbf{W} = \left\{ \begin{bmatrix} 3a + b \\ 4 \\ a - 5b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Is W a vector space? Why or why not?

Homework for Section 16

1. For the following matrix A, find an explicit description of $NulA$ by listing the vectors that span the null space.

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Find A such that the given set is $ColA$

$$\left\{ \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} \quad b, c, d \ \in \mathbb{R} \right\}$$

3. Let $\mathbf{A} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Is $\mathbf{w}$ in the Col A? Is $\mathbf{w}$ in the Nul A?

4. Define $T : \mathbb{P}_2 \longrightarrow \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$. For example, if $p(t) = 3 + 5t + 7t^2$ then $T(p) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$. Show that T is a linear transformation by taking arbitrary polynomials p and q in $\mathbb{P}_2$ and computing $T(p+q)$ and $T(cp)$.

Homework for Section 17

none