MATH 151 **Exam 3 Notes** **Spring**

1 Series

A series or *infinite series* is when you add the terms of an infinite sequence. That is,

$$a_1 + a_2 + a_3 + \ \dots \ + a_n + \ \dots$$

which is denoted

$$\sum_{n=1}^{\infty} a_n$$

or

$$\sum a_n$$

ex 1

$$1 + 2 + 3 + \ \dots \ = \ ?$$

ex 2 How about

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ \dots \ + \frac{1}{2^n} + \ \dots$$

If you add the first 2 terms you get $\frac{3}{4}$

If you add the first 3 terms you get $\frac{7}{8}$

If you add the first 5 terms you get $\frac{31}{32}$

It sure seems like

$$\sum \frac{1}{2^n} \ = \ 1$$

- This is the basic idea of how to determine if a series has a sum

Let's consider what are called ***partial sums***

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \end{aligned}$$

and in general

$$s_n = a_1 + a_2 + \ \ldots \ + a_n = \sum_{i=1}^{n} a_i$$

these partial sums form a NEW SEQUENCE that may or may not have a limit

If

$$\lim_{n \to \infty} s_n = s$$

exists **as a finite number** then s is the sum of the infinite series

$$\sum a_n$$

so $a_1, a_2, a_3, \ldots$ is a ***sequence***

$a_1 + a_2 + a_3 + \ldots = \sum a_n$ is a ***series***

$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$ is the series's ***nth partial sum*** the sequence $\{s_n\} = s_1, s_2, s_3, \ldots$ is a sequence of ***partial sums***

Definition

If the sequence $\{s_n\}$ is *convergent* and

$$\lim_{n\to\infty} s_n = s$$

exists as a real number, then the series

$$\sum a_n$$

is convergent and

$$\sum a_n = \mathbf{s}$$

Otherwise the series is divergent.

$$\sum a_n = \mathbf{s}$$

means that by adding sufficiently many terms of the series, we can get as close as we like to $\mathbf{s}$. Sound familiar?

One particular type of series is called a ***geometric series*** That is,

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$$

provided $a \neq 0$

each term is obtained from the preceding one by multiplying by a common ratio r.

If $r = 1$ then the series diverges.

If $-1 < r < 1$, then

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1} \tag{1}$$

multiplying both sides by r yields

$$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n \tag{2}$$

subtracting (2) from (1) gives us

$$s_n - rs_n = a - ar^n$$

since the middle terms cancel each other out

thus

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

and

$$\lim_{n \to \infty} s_n = \frac{a}{1 - r}$$

so for a geometric series

$$\sum ar^{n-1} = \begin{cases} \mathbf{C}, & if \ \mid r \mid \ < 1 \\ \mathbf{D}, & if \ \mid r \mid \ \geq 1 \end{cases}$$

If the geometric series converges then its sum is

$$\sum ar^{n-1} = \frac{a}{1 - r}$$

Geometric Series are one of the few types of series where if they converge you can actually calculate the sum. This is usually NOT the case.

ex 3 converge or diverge?

$$\sum \frac{3}{2^{n-1}}$$

$$\sum \frac{3}{2^{n-1}} = \sum 3 \left(\frac{1}{2}\right)^{n-1}$$

where $r = \frac{1}{2}$ and $a = 3$

Thus this sum converges to

$$\frac{3}{1 - \frac{1}{2}} = 6$$

ex 4 How about

$$1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \ \ldots$$

Here $a = 1$ and $r = \frac{3}{2}$ so the series diverges

ex 5 Find the sum of

$$\sum \frac{2}{4n^2 - 1}$$

Using partial fractions we get that

$$\frac{2}{4n^2 - 1} = \frac{1}{2n - 1} - \frac{1}{2n + 1}$$

so

$$s_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ \ldots + \left(\frac{1}{2n - 1} - \frac{1}{2n + 1}\right)$$

$$= 1 - \frac{1}{2n + 1} \implies \lim_{n \to \infty} s_n = 1$$

This type of series is called a **Telescoping Series** since the series collapses to only a few terms.

This is the other type of series where not only convergence or divergence is possible but also the sum.

Use this to your advantage. If the directions say to find the actual sum what does that mean?

ex 6 converge or diverge?

$$\sum \frac{1}{n}$$

This is an example of a ***harmonic series***

Theorem

If the series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Note that with any series you actually get 2 sequences

1. $\{s_n\}$ which is the sequence of partial sums and
2. $\{a_n\}$ which is the sequence of terms

Hopefully this makes sense. If a sum converges you must be adding very small pieces as $n \to \infty$, thus $\lim_{n\to\infty} (terms) = 0$

Another way to think about this is the contrapositive, that is the **Test for Divergence**

$$\text{If } \lim_{n\to\infty} a_n \text{ DNE or } \lim_{n\to\infty} a_n \neq 0 \implies \sum a_n \text{ diverges}$$

BE CAREFUL!!

If $\lim_{n\to\infty} a_n = 0$ what does that say about $\sum a_n$? NOTHING!

$$\lim_{n\to\infty} \frac{1}{2^n} = 0 \quad \textit{and} \quad \sum \frac{1}{2^n} \quad \textit{converges to } 1$$

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \textit{but} \quad \sum \frac{1}{n} \quad \textit{diverges}$$

ex 7 converge or diverge?

$$\sum \frac{n^2}{5n^2+4}$$

Note that

$$\lim_{n\to\infty} a_n = \frac{1}{5} \neq 0$$

Thus this sum diverges

Theorem

If $\sum a_n$ and $\sum b_n$ are convergent then

1. $\sum c\, a_n = c \sum a_n$
2. $\sum (a_n - b_n) = \sum a_n - \sum b_n$
3. $\sum (a_n + b_n) = \sum a_n + \sum b_n$

Worksheet for Section 1

1. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$

2. Show that the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n^2+1}$ diverges.

Homework for Section 1

1. Determine whether the geometric series is C or D. If C, find the sum.

   (a) $3 + 2 + \frac{4}{3} + \frac{8}{9} + \ldots$

   (b) $3 - 4 + \frac{16}{3} - \frac{64}{9} + \ldots$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$

2. Determine whether the series is C or D. If C, find the sum.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^2 - 1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1 + 2^n}{3^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{2}$

   (d) $\displaystyle\sum_{n=1}^{\infty} ln \left(\frac{n^2 + 1}{2n^2 + 1}\right)$

   (e) $\displaystyle\sum_{n=1}^{\infty} arctan\ n$

   (f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right)$

3. Determine whether the telescoping series is C or D. If C, find the sum.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

4. Find the values of x for which the series converges. Then find the sum for those values.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} 4^n\ x^n$

5. The **Cantor Set** is constructed as follows. Start with the closed interval $[0,1]$ and remove the open interval $(\frac{1}{3}, \frac{2}{3})$. That leaves the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Remove the open middle third of each of those. Continue with this process indefinitely. The Cantor Set consists of those numbers that remain after this process, $0, \frac{1}{3}, \frac{2}{3}, 1, \ldots$.

   (a) How many numbers are in the Cantor Set?

   (b) Show that the total length of all of those numbers removed is 1.

2 The Integral Test

Usually it is very difficult if not impossible to determine the exact sum of a series

These next few sections will develop several tests to check for convergence or divergence without finding the sums

ex 8 Find

$$\sum \frac{1}{n^2}$$

$$\sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ \ldots$$

There is no simple formula for the sum here

Consider the curve $y = 1/x^2$

Sketch the graph as well as some rectangles whose widths are one and heights are right endpoints. What do you notice?

Excluding the first rectangle, the area of the remaining rectangles is LESS THAN the area under the curve $y = 1/x^2$ for $x \geq 1$

So, the partial sums are LESS THAN $\quad 1 + \displaystyle\int_1^\infty 1/x^2 \ dx \ = \ 2$

$$\Longrightarrow \quad \sum \frac{1}{n^2} \ < \ 2$$

In a similar fashion we can use the rectangles above a curve to show divergence

So,

Integral Test

Suppose that f is a continuous, positive and decreasing function on $[1, \infty)$. Then

1. If $\displaystyle\int_1^\infty f(x)\, dx$ is convergent $\Longrightarrow \quad \sum a_n$ is convergent.
2. If $\displaystyle\int_1^\infty f(x)\, dx$ is divergent $\Longrightarrow \quad \sum a_n$ is divergent.

Note that you do NOT have to start the series at $n = 1$

To test

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$$

Use

$$\int_4^\infty \frac{1}{(x-3)^2}\, dx$$

ex 9 test

$$\sum \frac{n}{n^2+1}$$

Look at

$$\int_1^\infty \frac{x}{x^2+1}\, dx \;=\; \frac{1}{2}\int_1^\infty \frac{1}{u}\, du \;=\; \ldots \;=\; \infty$$

thus the series diverges

ex 10 test

$$\sum \frac{1}{n^2+1}$$

Look at

$$\int_1^\infty \frac{1}{x^2+1}\,dx \;=\; \ldots \;=\; arctan\ x \;=\; \ldots\ \frac{\pi}{4} \;\Longrightarrow\; \mathbf{C}$$

Realize that the series does NOT equal $\pi/4$ we only know that it converges

There is a special series called a ***p-series***, that is $\sum \frac{1}{n^p}$

$$\sum \frac{1}{n^p} = \begin{cases} \mathbf{C},\ \ if\ p\ > 1 \\ \mathbf{D},\ \ if\ p\ \le 1 \end{cases}$$

Again, the integral test does NOT tell you the sum only convergence or divergence

$$\sum \frac{1}{n^2} \;=\; \frac{\pi^2}{6}$$

However

$$\int_1^\infty \frac{1}{x^2}\ dx \;=\; 1$$

ex 11 converge or diverge

$$\sum \frac{ln\ n}{n}$$

The function $\frac{ln\ x}{x}$ is positive and continuous but is it decreasing? Note that

$$f'(x) = \frac{1 - ln\ x}{x^2} \quad \Longrightarrow \quad f\ is\ decreasing\ for \quad x\ >\ e$$

So the integral test yields

$$\int_1^\infty \frac{ln\ x}{x}\,dx \ = \ \int_1^\infty u\,du \ = \ \lim_{t\to\infty} \frac{(ln\ t)^2}{2} \ = \ \ldots \ = \infty$$

thus the series diverges

Worksheet for Section 2

1. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$

2. Determine whether the series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$

Homework for Section 2

1. Determine whether the series is C or D.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$

   (b) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{0.85}}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

   (g) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ ln\ n}$

3 Comparison Tests

This is the exact same concept as what we covered before.

Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with *positive terms*

1. If $\sum b_n$ is convergent **AND** $a_n \leq b_n \quad \forall\, n \quad \Longrightarrow \quad \sum a_n$ is convergent
2. If $\sum b_n$ is divergent **AND** $a_n \geq b_n \quad \forall\, n \quad \Longrightarrow \quad \sum a_n$ is divergent

ex 12 Determine the convergence or divergence of

$$\sum \frac{1}{2+3^n}$$

Obviously if you use the comparison theorem you will need a series to compare the given one to.

For this example the given series looks a lot like...

$$\sum \frac{1}{3^n}$$

Why pick this one?

Because it is a convergent geometric series. But now you need the appropriate comparison. Since our series converges you need to make sure that the given series is $\leq$ our convergent geometric series.

Since

$$\frac{1}{2+3^n} < \frac{1}{3^n} \implies \sum \frac{1}{2+3^n} \quad converges$$

ex 13 Determine the convergence or divergence of

$$\sum \frac{ln\ n}{n}$$

We did a similar problem previously. Note that

$$\frac{ln\ n}{n} > \frac{1}{n}\ for\ n \geq 3 \implies \sum \frac{ln\ n}{n} \quad \text{diverges}$$

Often a series closely resembles a p-series or a geometric series but you are unable to determine the appropriate comparison.

ex 14

$$\sum \frac{1}{2^n - 1}$$

Note that

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

So here a straight comparison will not work. Under such circumstances use the...

Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with *positive terms*

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \text{ with } c > 0 \text{ and } c \text{ a finite number}$$

Then either **BOTH** series converge OR **BOTH** series diverge.

That is, BOTH do the same thing.

ex 15 Test

$$\sum \frac{\sqrt{n}}{n^2+1}$$

- LCT works very well comparing a messy algebraic series to a p-series
- when choosing a comparison series focus on the highest powers of n in the top and bottom

Focus on

$$\sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$$

Which is a convergent p-series

So

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2+1}\right)\left(\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \implies \mathbf{C}$$

Given Series	**Comparison Series**	**Conclusion**
$\sum \frac{1}{3n^2 - 4n + 5}$	$\sum \frac{1}{n^2}$	both converge
$\sum \frac{1}{\sqrt{3n-2}}$	$\sum \frac{1}{\sqrt{n}}$	both diverge
$\sum \frac{n^2 - 10}{4n^5 + n^3}$	$\sum \frac{1}{n^3}$	both converge

ex 16 Test

$$\sum \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$$

Compare with

$$\sum \frac{2n^2}{\sqrt{n^5}} = \sum \frac{2}{n^{1/2}}$$

Which is a divergent p-series

So

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{2n^2 + 3n}{\sqrt{5 + n^5}}\right)\left(\frac{n^{1/2}}{2}\right) = \lim_{n\to\infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}}$$

$$= \lim_{n\to\infty} \frac{2 + \frac{3}{n}}{2\sqrt{\frac{5}{n^5} + 1}} = 1 \implies \mathbf{D}$$

Worksheet for Section 3

1. Determine whether the series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{ln\ n}{n^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} ln\left(\frac{n}{2n+5}\right)$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{cos^2\ n}{n^2+1}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

Homework for Section 3

1. Determine whether the series converges or diverges.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{cos^2\, n}{n^2+1}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n\ 4^n}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{arctan\ n}{n^{1.2}}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$

2. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is convergent. Show that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

then $\sum a_n$ is also convergent. Use this fact to show the following converges:

$$\sum_{n=1}^{\infty} \frac{ln\ n}{n^3}$$

3. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is divergent. Show that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

then $\sum a_n$ is also divergent. Use this fact to show the following diverges:

$$\sum_{n=2}^{\infty} \frac{1}{ln\ n}$$

4 Alternating Series

An *alternating series* is a series whose terms alternate parity, that is alternate sign.

Alternating Series Test

If the alternating series

$$\sum (-1)^{n-1}\, b_n \;=\; b_1 - b_2 + b_3 - b_4 + \;\ldots \qquad with\;\; b_n > 0$$

satisfies:

1. $$b_{n+1} < b_n \;\; \forall \;\; n$$

2. $$\lim_{n\to\infty} \; b_n \;=\; 0$$

then the series is convergent.

ex 17 Test the following series for convergence

$$\sum \; (-1)^{n-1}\, \frac{1}{n} \;=\; 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \;\ldots$$

Since

$$\frac{1}{n+1} \;<\; \frac{1}{n} \quad\Longrightarrow\quad b_{n+1} < b_n$$

and

$$\lim_{n\to\infty}\; b_n \;=\; \lim_{n\to\infty}\; \frac{1}{n} \;=\; 0 \quad\Longrightarrow\quad \mathbf{C}$$

ex 18 Test the following series for convergence

$$\sum \frac{(-1)^{n+1}(n+1)}{n}$$

This series passes condition (1), however

$$\lim_{n\to\infty} \frac{n+1}{n} \;=\; 1 \;\neq\; 0 \quad \Longrightarrow \quad \mathbf{D}$$

ex 19 Test the following

$$\sum (-1)^{n+1} \frac{n^2}{n^3+1}$$

$$Is \quad \frac{n^2}{n^3+1} \quad decreasing?$$

$$Consider \quad f(x) = \frac{x^2}{x^3+1} \quad \Longrightarrow$$

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} \quad \Longrightarrow \quad f'(x) < 0 \;\; on \;\; (\sqrt[3]{2}, \infty)$$

$$Also \quad \lim_{n\to\infty} \frac{n^2}{n^3+1} \;=\; 0 \quad \Longrightarrow \quad \mathbf{C}$$

Alternating Series Estimation Theorem

If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies:

1. $0 \leq b_{n+1} \leq b_n$ and
2. $\lim_{n\to\infty} b_n = 0$

Then

$$| R_n | = | s - s_n | \leq b_{n+1}$$

that is, the size of the error in the sum is LESS THAN the absolute value of the first neglected term

ex 20 Find the error involved in approximating the following sum by its first 6 terms. That is, find R_6

$$\sum (-1)^{n+1} \left(\frac{1}{n!}\right)$$

Using the preceding theorem we know that

$$| R_6 | \leq b_7 \quad and \quad b_7 = \frac{1}{7!} = \frac{1}{5040} \approx .0002$$

- **BEWARE** as this ONLY applies to alternating series that meet the theorem's conditions

Worksheet for Section 4

1. Test the series for convergence or divergence.

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+\sqrt{n}}$

Homework for Section 4

1. Determine whether the series is C or D.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$

   (d) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{ln\ n}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{cos(n\pi)}{n^{3/4}}$

2. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ is convergent. How many terms do you need to add so that $|$ error $| < 0.00005$?

5 Absolute Convergence

Definition

A series $\sum a_n$ is *absolutely convergent* if the series of absolute values,

$\sum |a_n|$ is convergent.

ex 21 Determine the type of convergence for the alternating harmonic series

$$\sum \frac{(-1)^{n-1}}{n}$$

The alternating harmonic series

$$\sum \frac{(-1)^{n-1}}{n}$$

satisfies the AST, the alternate series test, however

$$\sum \left|\frac{(-1)^{n-1}}{n}\right| = \sum \frac{1}{n} \implies \mathbf{D}$$

1. $\sum a_n$ is ***absolutely convergent, AC,*** if $\sum |a_n|$ converges

2. $\sum a_n$ is ***conditionally convergent, CC,*** if $\sum a_n$ converges but $\sum |a_n|$ diverges

ex 22 Determine if the series is **AC**, **CC** or **D**

$$\sum \frac{(-1)^n}{\sqrt{n}}$$

by the AST $\longrightarrow$ **C**

the absolute value of the series is a divergent p-series

$\implies$ **CC**

ex 23 Determine if the series is **AC**, **CC** or **D**

$$\sum \frac{(-1)^{n-1}}{n^2}$$

the absolute value of the series, $\sum \left|\frac{(-1)^{n-1}}{n^2}\right|$ is a convergent p-series

$\implies$ **AC**

Combining comparison tests and geometric series we derive 2 more tests

The Ratio Test

This tests for absolute convergence

1. If
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \implies \sum a_n \ \ is \ \ \mathbf{AC}$$

2. If
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \ \ or \ \ \infty \implies \sum a_n \ \ is \ \ \mathbf{D}$$

3. If
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \implies no \ \ conclusion$$

ex 24 Test

$$\sum \frac{n^2\ 2^{n+1}}{3^n}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[(n+1)^2 \left(\frac{2^{n+2}}{3^{n+1}}\right)\left(\frac{3^n}{n^2\ 2^{n+1}}\right)\right]$$

$$= \lim_{n\to\infty} \frac{2(n+1)^2}{3n^2} = \frac{2}{3} \implies \mathbf{AC}$$

ex 25 Test

$$\sum \frac{n^n}{n!}$$

$$\lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{(n+1)!}\ \frac{n!}{n^n}\right] = \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{n+1}\ \frac{1}{n^n}\right]$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \implies \mathbf{D}$$

The Root Test

This works well with nth powers

1. If
$$\lim_{n\to\infty} \sqrt[n]{|\ a_n\ |} \ < \ 1 \quad \Longrightarrow \quad \sum a_n \ \ is \ \ \mathbf{AC}$$
2. If
$$\lim_{n\to\infty} \sqrt[n]{|\ a_n\ |} \ > \ 1 \ \ or \ \ \infty \quad \Longrightarrow \quad \sum a_n \ \ is \ \ \mathbf{D}$$
3. If
$$\lim_{n\to\infty} \sqrt[n]{|\ a_n\ |} \ = \ 1 \quad \Longrightarrow \quad \text{no conclusion}$$

ex 26 Test
$$\sum \frac{e^{2n}}{n^n}$$
$$\lim_{n\to\infty} \sqrt[n]{|\ a_n\ |} \ = \ \lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} \ = \ \lim_{n\to\infty} \frac{e^2}{n} \ = \ 0 \quad \Longrightarrow \quad \mathbf{AC}$$

ex 27 Test
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(ln\ n)^n}$$
$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(ln\ n)^n}\right|} \ = \ \lim_{n\to\infty} \sqrt[n]{\left(\frac{1^n}{(ln\ n)^n}\right)} \ = \ \lim_{n\to\infty} \frac{1}{ln\ n} \ = \ 0 \ \Longrightarrow \ \mathbf{AC}$$

Worksheet for Section 5

1. Determine whether the series is absolutely convergent, conditionally convergent or divergent.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+3)^n}{(3n+2)^n}$

Homework for Section 5

1. Determine whether the series is AC, CC or D.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ arctan\ n}{n^2}$

   (f) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{ln\ n}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

6 Strategies

The following 5-step process might help when evaluating series:

1. Does $\lim_{n\to\infty} a_n = 0$? If not the series diverges.

2. Can you classify the series?
   (a) Geometric ?
   (b) Telescoping ?
   (c) P-series ?
   (d) Alternating ?

3. Is the associated function easy to integrate? Then use the Integral Test.

4. Is the series of the form $(a_n)^n$? Then use the Root Test.

5. Does the series involve factorials, constants to the n^{th} power, products? Use the Ratio Test.

Let's look at the following examples to identify the appropriate test

1. $\sum \dfrac{n+1}{3n+1}$ *look at* $\lim\limits_{n\to\infty} a_n \quad \longrightarrow \quad$ **D**

2. $\sum \left(\dfrac{\pi}{6}\right)^n$ *geometric* $\longrightarrow$ **C**

3. $\sum n\, e^{-n^2}$ *integral test* $\longrightarrow$ **C**

4. $\sum \dfrac{1}{3n+1}$ *limit comparison test* $\longrightarrow$ **D**

5. $\sum \dfrac{(-1)^n\, 3}{4n+1}$ *alternating series* $\longrightarrow$ **C**

6. $\sum \dfrac{n!}{10^n}$ *ratio test* $\longrightarrow$ **D**

7. $\sum \left(\dfrac{n+1}{2n+1}\right)^n$ *root test* $\longrightarrow$ **C**

Worksheet for Section 6

1. Determine whether the series is absolutely convergent, conditionally convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

   (d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

   (e) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Homework for Section 6

1. Determine whether the series is C or D.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2\ 2^{n-1}}{(-5)^n}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{ln\ n}}$

(e) $\displaystyle\sum_{n=1}^{\infty} n^2\ e^{-n}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n\ n^2}{n!}$

(g) $\displaystyle\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$