1 Coordinate Systems

Theorem 1

Let $B = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. For each $x \in V$ there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$x = c_1 b_1 + \ldots + c_n b_n$$

Definition 1

Suppose $B = \{b_1, \ldots, b_n\}$ is a basis for a vector space V and $x \in V$, *the coordinates of x relative to the basis B* are the weights $c_1, \ldots, c_n$ such that $x = c_1 b_1 + \ldots + c_n b_n$

so

$$[x]_B = \begin{bmatrix} c_1 \\ . \\ . \\ . \\ c_n \end{bmatrix}$$

is the coordinate vector of x relative to B

If a basis $B = \{b_1, \ldots, b_n\}$ and $P_B = [b_1 \ldots b_n]$ then $x = c_1 b_1 + \ldots + c_n b_n$ is equivalent to

$$x = P_B [x]_B$$

and P_B is called the *change of coordinates matrix*

Note that if the columns of P_B are a basis for $\mathbb{R}^n \implies$

P_B in invertible $\implies$

the mapping is a one to one linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$ by the IMT

Theorem 2

Suppose $B = \{b_1, \ldots, b_n\}$ is a basis for a vector space V. Then the coordinate mapping $x \longrightarrow [x]_B$ is a one to one linear transformation from V onto $\mathbb{R}^n$

This one to one linear transformation is called an *isomorphism*

ex 1

Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$

1. Is $B = \{b_1, b_2, b_3\}$ a basis for $\mathbb{R}^3$?
2. Find the change of coordinates matrix
3. Find $[x]_B$ for the given x

1.

$$B = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix} \implies ?$$

2.

$$B = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

3.

$$P_B x = \begin{bmatrix} 1 & -3 & 3 & -8 \\ 0 & 4 & -6 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \implies$$

$$[x]_B = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

Worksheet for Section 1

1. Find the vector $\mathbf{x}$ determined by the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$ and the basis
$\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$

2. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

3. The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 3 + t - 6t^2$ relative to $\mathcal{B}$.

Homework for Section 1

1. Use coordinate vectors to determine the linear independence of the following set of polynomials.

$$1 - 2t^2 - 3t^3 \ , \ t + t^3 \ , \ 1 + 3t - 2t^2$$

2 The Dimension of a Vector Space

Theorem 3

If a vector space V has a basis $B = \{b_1, \dots, b_n\}$ then any set in V containing more than n vectors is *linearly dependent*

Theorem 4

If a vector space V has a basis of n vectors then every basis of V contains exactly n vectors

Definition 2

If V is spanned by a finite set, V is finite dimensional and the dimension of V, denoted $dimV$, is the number of vectors in a basis for V. The dimension of the zero vector space is 0.

ex 2

$$dim\mathbb{R}^n \;=\; ?$$

$$dim\mathbb{P}_n \;=\; ?$$

ex 3

Find a basis and state the dimension.

$$\left\{ \begin{bmatrix} 4s \\ -3s \\ t \end{bmatrix} \ : \ s, t \ in \ \mathbb{R} \right\}$$

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} , \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

the basis is $\{v_1, v_2\}$ and $dim = 2$

Theorem 5

Let H be a subspace of a finite dimensional vector space V. Any linearly independent set in H can be expanded to a basis for H.

Also, $dimH \leq dimV$

Theorem 6

Basis Theorem

Let V be a p-dimensional vector space with $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

In other words, if a set has the right number of elements, you only need to show the set is either linearly independent or that the set spans the space. Sometimes linear independence is much easier to show.

Recall that the dimension of the $NulA$ is the number of free variables in the equation $Ax = 0$ and the dimension of the $ColA$ is the number of pivot columns in A.

Worksheet for Section 2

1. Find a basis and state the dimension.

$$\left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} \quad : \; a, b, c \; in \; \mathbb{R} \right\}$$

2. Find the dimensions of Nul A and Col A.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. The first four Laguerre polynomials are $1, 1-t, 2-4t+t^2$ and $6-18t+9t^2-t^3$. Show that these polynomials form a basis for $\mathbb{P}_3$.

Homework for Section 2

1. Determine the dimensions of $Nul\ A$ and $Col\ A$ for:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Let B be the basis of $\mathbb{P}_2$ consisting of the first three Laguerre polynomials which are $1, 1-t, 2-4t+t^2$. Let $p(t) = 7 - 8t + 3t^2$. Find the coordinate vector of p relative to B.

3 Rank

Given an $m \times n$ matrix A, the set of all combinations of row vectors is called the row space of A, denoted $Row A$

Note that $Row A$ is a subspace of $\mathbb{R}^n$ and $Row A = Col A^T$

Theorem 7

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A *as well as* B.

ex 4

Find a basis for $Row A$, $Col A$ and $Nul A$

$$\mathbf{A} = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & 2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{B}$$

So a basis for $Row A$ and $Row B$ is:

$$\{(1,3,-5,1,5),(0,1,2,2,-7),(0,0,0,-4,20)\}$$

1,2 and 4 are pivot columns so columns 1,2 and 4 of A form the basis for $ColA$

$$ColA \;=\; \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix} , \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

We need RREF for $NulA$ ...

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so

$$x_1 = -x_3 - x_5$$
$$x_2 = 2x_3 - 3x_5$$
$$x_4 = 5x_5$$

$\Longrightarrow \quad x_3, x_5$ are free $\Longrightarrow$

$$\text{Basis for } NulA \;=\; \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Definition 3

The *Rank* of A is the dimension of the column space of A

Rank Theorem

Given an $m \times n$ matrix A, $dim\ ColA = dim\ RowA$

This common dimension, the Rank of A, also equals the number of pivot positions in A and satisfies:

$$rank\ A + dim\ NulA = n$$

That is:

$$\text{(pivot columns)} + \text{(nonpivot columns)} = \text{(number of columns)}$$

IMT continued...

m. columns of A form a basis for $\mathbb{R}^n$
n. $ColA = \mathbb{R}^n$
o. $dim\ ColA = n$
p. $rank\ A = n$
q. $NulA = 0$
r. $dim\ NulA = 0$

Worksheet for Section 3

1. The matrices A and B are row equivalent. Find the following:

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) rank A

   (b) dim Nul A

   (c) basis for Col A

   (d) basis for Row A

   (e) basis for Nul A

2. Suppose a 5×6 matrix A has four pivot columns. What is the dim Nul A? Is Col $A = \mathbb{R}^4$? Why or why not?

Homework for Section 3

1. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Find

   (a) rank A

   (b) dim Nul A

   (c) a basis for $Col\ A$

   (d) a basis for $Row\ A$

   (e) a basis for $Nul\ A$

2. If A is a 4×3 matrix, what is the largest possible dimension of the row space of A? If A is a 3×4 matrix, what is the largest possible dimension of the row space of A? Explain.

4 Eigenvectors and Eigenvalues

Definition 4

An *eigenvector* of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ, called an *eigenvalue* of A.

Although an eigenvector can not be zero, an eigenvalue can.

Also, λ is an eigenvalue if $(A - \lambda I)x = 0$ has a nontrivial solution.

The set of all solutions is called the *eigenspace*

ex 5

$$\text{If } \mathbf{A} = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}, \ \lambda = 4$$

Find a basis for the eigenspace corresponding to $\lambda = 4$

So,

$$A - 4I \ = \ \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \ = \ \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$$

The augmented matrix for $(A - 4I)x = 0$ is

$$\begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -9/6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $x_1 = \frac{3}{2}x_2$ and x_2 is free. Therefore a basis for the eigenspace is

$$\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

Theorem 8

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Why do you think that is?

Consider the 2 by 2 case...

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ 0 & a_{22} - \lambda \end{bmatrix}$$

$$\lambda \text{ is an eigenvalue } \iff (A - \lambda I)x = 0 \text{ has a nontrivial solution}$$
$$\iff \text{ there are free variables } \iff \text{ the entries on the diagonal are } 0$$
$$\iff \lambda \text{ is a diagonal entry}$$

Theorem 9

If $v_1, v_2, \ldots, v_r$ are eigenvectors corresponding to *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of an $n \times n$ matrix $A \Longrightarrow \{v_1, v_2, \ldots, v_r\}$ are linearly independent.

Worksheet for Section 4

1. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the eigenvalue.

2. Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

3. Find a basis for the eigenspace corresponding to each eigenvalue.

$$\mathbf{A} = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \quad \lambda = 1, 5$$

Homework for Section 4

1. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

   Find a basis for the eigenspace corresponding to $\lambda = 4$

2. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

5 The Characteristic Equation

ex 6

Find the eigenvalues of $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

In other words, find all λ such that $(A - \lambda I)x = 0$ has a nontrivial solution.

Or, find all λ such that $A - \lambda I$ is NOT invertible.

Now, $(A - \lambda I) = \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$

and that is not invertible if the $det = 0$

So,

$$det(A - \lambda I) = (2-\lambda)(2-\lambda) - (7)(7) = (\lambda^2 - 4\lambda - 45)^*$$

$$\lambda^2 - 4\lambda - 45 = 0 \Longleftrightarrow (\lambda - 9)(\lambda + 5) = 0 \Longleftrightarrow \lambda = 9, -5$$

Thus, the eigenvalues of A are 9 and -5.

$*$ is called the **characteristic polynomial** of A

BE CAREFUL:

Row reduction is used to find eigenvectors, it can **NOT** be used to find eigenvalues since the echelon form of a matrix A usually does not display the eigenvalues of A.

Worksheet for Section 5

1. Find the characteristic polynomial and the eigenvalues of $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

2. Find the characteristic polynomial of $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Homework for Section 5

none

6 Diagonalization

Sometimes it is extremely useful when a matrix A is in another form, $A = PDP^{-1}$ where D is a diagonal matrix. Why? Diagonal matrices are very easy to deal with.

ex 7

If

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

find D^2, D^3, ...

In general

$$\mathbf{D}^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}$$

$$\implies \quad A^k \text{ is also easy to compute since } \; A^k = PD^kP^{-1}$$

Definition 5

A square matrix A is DIAGONALIZABLE if $A = PDP^{-1}$ for some invertible matrix P and diagonal matrix D.

Theorem 10

An $n \times n$ matrix A is diagonalizable $\iff$ A has n linearly independent eigenvectors.

$A = PDP^{-1} \iff$ the columns of P are the n linearly independent eigenvectors of A. In this case the diagonal entries of D are the eigenvalues of A that correspond to the eigenvectors in P.

Here is the 4 step process to diagonalize a square matrix A

1. Find the eigenvalues of A
2. Find the n linearly independent eigenvectors of A (If this fails, A is NOT diagonalizable)
3. Construct P from step 2. The order is unimportant.
4. Construct D from step 1. The order of the eigenvalues MUST match the order chosen for the columns of P

Theorem 11

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

ex 8

If the eigenvalues are $\lambda = 1, 2, 3$ diagonalize the following matrix:

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

for $\lambda = \mathbf{3}$ we have $(A - 3I) = \begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\implies$ the solution is $x_3 \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix}$ and the basis is $\mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

for $\lambda = \mathbf{2}$ we have $(A - 2I) = \begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\implies$ the solution is $x_3 \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix}$ and the basis is $\mathbf{v_2} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

for $\lambda = \mathbf{1}$ we have $(A - 1I) = \begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix} \sim \ldots$

$\implies$ the solution is $x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the basis is $\mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

So

$$\mathbf{P} = [v_1\ v_2\ v_3] = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ex 9

Diagonalize the following matrix, if possible:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

The characteristic polynomial is $-(\lambda - 1)(\lambda + 2)^2$, thus the eigenvalues are $\lambda = 1$ and $\lambda = -2$, however

$$\text{basis for } \lambda = 1 : \qquad \mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{basis for } \lambda = -2 : \qquad \mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Thus, A is NOT diagonalizable. Why?

Worksheet for Section 6

1. Let $A = PDP^{-1}$ and compute A^4 if:

$$\mathbf{P} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

2. If the eigenvalues are $\lambda = 2, 8$ diagonalize the following matrix:

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Homework for Section 6

none

7 Eigenvectors and Linear Transformations

Basically if V is an $n-$ *dimensional* vector space and W is an $m-$ *dimensional* vector space, then the linear transformation T sends vectors, $x \in V$, from V to W.

Now $[x]_B$ is the coordinate vector from V and $[T(x)]_C$ is the related coordinate vector from W. What is the connection that sends $[x]_B \longrightarrow [T(x)]_C$?

Well if the matrix M is the matrix representation of T, then as far as coordinate vectors go, the action of T on x can be viewed as *left-multiplication* by M.

ex 10

Let $T : \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ be defined by $T(a + bt + ct^2) = b + 2ct$

Recognize T?

Find the B matrix for T when the basis is $\{1, t, t^2\}$

First, find the images of the basis vectors:

$$T(1) = 0\ ,\quad T(t) = 1\ ,\quad T(t^2) = 2t$$

So

$$[T(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(t)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } [T(t^2)]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Thus the B matrix, $[T]_B$, is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A linear transformation usually appears as a matrix transformation: $x \longrightarrow Ax$

If A is diagonalizable, there is a basis B consisting of eigenvectors of A. That is, the B matrix is diagonal.

Theorem 12

Suppose $A = PDP^{-1}$, D a diagonal $n \times n$ matrix. If B is a basis for $\mathbb{R}^n$ formed from the columns of P, then D is the B matrix for the transformation $x \longrightarrow Ax$

Worksheet for Section 7

1. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \longmapsto W$ be a linear transformation with the property that:

   $$T(\mathbf{d}_1) = 2\mathbf{b}_1 - 3\mathbf{b}_2\ , \qquad T(\mathbf{d}_2) = -4\mathbf{b}_1 + 5\mathbf{b}_2$$

   Find the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$

2. Let $T : \mathbb{P}_2 \longmapsto \mathbb{P}_3$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $(t+5)\mathbf{p}(t)$.

   (a) Find the image of $\mathbf{p}(t) = 2 - t + t^2$

   (b) Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$

Homework for Section 7

none

8 Applications to Differential Equations

Can you recall the solution to the following differential equation?

$$\frac{dx}{dt} = Ax$$

The solution is $x(t) = Ae^{kt}$

We could verify this. So,

$$\frac{dx}{dt} = Ax \iff x' = Ax$$

Let's look at the possible solution $x = ve^{\lambda t}$

$$\implies \lambda ve^{\lambda t} = Ave^{\lambda t} \implies \lambda v = Av$$

Does this look familiar?

ex 11

Find the general solution to the following system:

$$(x_1)' = 4x_1 + 2x_2$$

$$(x_2)' = 3x_1 - x_2$$

Thus

$$x' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} x$$

and the characteristic equation of the coefficient matrix is

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} x \;=\; \lambda^2 - 3\lambda - 10 \;=\; (x+2)(x-5)$$
$$\Longrightarrow \text{ the eigenvalues are } \; \lambda = -2, 5$$

So for $\lambda = -2$ solve

$$\begin{bmatrix} 6 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix} \text{ and get the eigenvector } \mathbf{v_1} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

and for $\lambda = 5$ solve

$$\begin{bmatrix} -1 & 2 & 0 \\ 3 & -6 & 0 \end{bmatrix} \text{ and get the eigenvector } \mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So the general solution of the system is

$$x(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$$

Worksheet for Section 8

none

Homework for Section 8

none

9 Inner Product Length and Orthogonality

Let u, v be vectors in $\mathbb{R}^n$. Then, the number $u^T v$ is an *inner product* or *dot product* denoted $\mathbf{u} \bullet \mathbf{v}$

Theorem 13

Let u, v and w be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

1. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \bullet \mathbf{w} = \mathbf{u} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{w}$
3. $(\mathbf{cu}) \bullet \mathbf{v} = c(\mathbf{u} \bullet \mathbf{v}) = \mathbf{u} \bullet (\mathbf{cv})$
4. $\mathbf{u} \bullet \mathbf{u} \geq 0$, and $\mathbf{u} \bullet \mathbf{u} = 0 \iff \mathbf{u} = \mathbf{0}$

The **length** or **norm** of a vector v is a non-negative scalar $\|v\|$ such that $\|v\| = \sqrt{v \bullet v} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$ and $\|v\|^2 = v \bullet v$

If Let $u, v \in \mathbb{R}^n$, then the distance between them is $dist(u, v) = \|u - v\|$

Definition 6

Two vectors $u, v \in \mathbb{R}^n$ are **orthogonal** if $u \bullet v = 0$

Theorem 14

Two vectors $u, v \in \mathbb{R}^n$ are orthogonal $\iff \left\|u+v\right\|^2 = \left\|u\right\|^2 + \left\|v\right\|^2$

Worksheet for Section 9

1. Given that $\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$ find $\mathbf{w} \bullet \mathbf{w}$, $\mathbf{x} \bullet \mathbf{w}$ and $\frac{\mathbf{x} \bullet \mathbf{w}}{\mathbf{w} \bullet \mathbf{w}}$

2. Find the distance between $\mathbf{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

3. Are $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$ orthogonal?

Homework for Section 9

1. Suppose a vector y is orthogonal to vectors u and v. Show that y is orthogonal to the vector $u + v$

10 Orthogonal Sets

A set of vectors $\{u_1, u_2, \ldots, u_p\}$ in $\mathbb{R}^n$ is an orthogonal set if each pair is orthogonal.

Theorem 15

If $S = \{u_1, u_2, \ldots, u_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and is a basis for the subspace spanned by S.

Definition 7

An *orthogonal basis* for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem 16

Let $\{u_1, u_2, \ldots, u_p\}$ be an orthogonal basis. For each $y \in W$

$y = c_1u_1 + c_2u_2 + \ldots + c_pu_p$ and the weights are given by $c_j = \dfrac{y \bullet u_j}{u_j \bullet u_j}$

A set of vectors $\{u_1, u_2, \ldots, u_p\}$ is an orthonormal set if it is an orthogonal set of unit vectors, that is, vectors of length 1.

Theorem 17

An $m \times n$ matrix U has orthonormal columns $\iff U^T U = I$

Worksheet for Section 10

1. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$ and express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

2. Determine if $\begin{bmatrix} -.6 \\ .8 \end{bmatrix}, \begin{bmatrix} .8 \\ .6 \end{bmatrix}$ are orthonormal.

Homework for Section 10

1. Is the following set of vectors orthonormal? If it is only orthogonal, then normalize them.

$$\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \quad \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$$

11 Orthogonal Projections

Given a vector y and W a subspace of $\mathbb{R}^n$, there exists a vector $\hat{y} \in W$ such that:

1. $\hat{y}$ is the unique vector in W for which $y - \hat{y}$ is orthogonal to W
2. $\hat{y}$ is the unique vector in W closest to y

Theorem 18

ORTHOGONAL DECOMPOSITION THEOREM

If W is a subspace of $\mathbb{R}^n$, then each $y \in \mathbb{R}^n$ can be written uniquely as

$$y = \hat{y} + z \text{ where } \hat{y} \in W$$

In fact, if $\{u_1, u_2, \ldots, u_p\}$ is any orthogonal basis of W, then

$$\hat{y} = \frac{y \bullet u_1}{u_1 \bullet u_1} u_1 + \ldots + \frac{y \bullet u_p}{u_p \bullet u_p} u_p \text{ and } z = y - \hat{y}$$

The vector $\hat{y}$ is called the orthogonal projection of y onto W, denoted $proj_W\ y$

Theorem 19

BEST APPROXIMATION THEOREM

If W is a subspace of $\mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\hat{y}$ the orthogonal projection of y onto W, then

$$\left\|y - \hat{y}\right\| \; < \; \left\|y - v\right\| \; \forall v \neq \hat{y} \in W$$

Theorem 20

If $\{u_1, u_2, \ldots, u_p\}$ is an orthonormal basis $\Longrightarrow$

$$proj_W \; y = \hat{y} = (y \bullet u_1)u_1 + (y \bullet u_2)u_2 + \ldots + (y \bullet u_p)u_p$$

and if $U = [u_1 u_2 \ldots u_p]$ then

$proj_W \; y = UU^T y \;\; \forall y \in \mathbb{R}^n$

Worksheet for Section 11

1. Given that $\mathbf{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ verify that $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthogonal set and find the orthogonal projection of $\mathbf{y}$.

2. Given that $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$ and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$ Let W be the subspace spanned by the $\mathbf{u}$'s and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

Homework for Section 11

none

12 Gram-Schmidt Process

This process is an algorithm for producing an orthogonal or orthonormal basis.

Given a basis $\{x_1, x_2, \ldots, x_p\}$ for a subspace W of $\mathbb{R}^n$ let:

$v_1 = x_1$

$v_2 = x_2 - \frac{x_2 \bullet v_1}{v_1 \bullet v_1} v_1$

$v_3 = x_3 - \frac{x_3 \bullet v_1}{v_1 \bullet v_1} v_1 - \frac{x_3 \bullet v_2}{v_2 \bullet v_2} v_2$

$\vdots$

$v_p = x_p - \frac{x_p \bullet v_1}{v_1 \bullet v_1} v_1 - \ldots - \frac{x_p \bullet v_{p-1}}{v_{p-1} \bullet v_{p-1}} v_{p-1}$

then $\{v_1, v_2, \ldots, v_p\}$ is an orthogonal basis for W

Theorem 21

QR FACTORIZATION THEOREM

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for $ColA$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

ex 12

Find a QR factorization of:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

If the columns of A are the vectors x_1, x_2, x_3, use Gram-Schmidt to find the orthogonal basis...

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Scale v_3 by multiplying by 3, then normalize the vectors to obtain the columns of Q

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$$

Now, $Q^T A = Q^T(QR) = IR = R$, so

$$R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

Worksheet for Section 12

1. Use the Gram-Schmidt process to produce an orthogonal basis for
$$\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$$

2. Find an orthonormal basis of the subspace spanned by the vectors in number 1 above.

Homework for Section 12

1. The following set of vectors is a basis for a subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W.

$$\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix}$$

13 Least Squares Problems

This is the same idea that we have spoken about last year in calculus. If you were unable to integrate a function, what could you do?

Now, what if $Ax = b$ has no solution, can we find an x that makes Ax as close as possible to b?

The *least squares solution*, denoted $\hat{x}$ is $\hat{x} \in \mathbb{R}^n$ such that

$$\left\|b - A\hat{x}\right\| \leq \left\|b - Ax\right\| \ \forall x \in \mathbb{R}^n$$

Think about points in a two dimensional plane. How would points not in that 2-d plane be closest to a vector that is in that 2-d plane?

we are looking for the *projections*

Theorem 22

The set of least squares solutions of $Ax = b$ coincides with the set of non-empty solutions to the normal equations, $A^T Ax = A^T b$

Theorem 23

A^TA is invertible $\iff$ the columns of A are linearly independent.
Then $Ax = b$ has only one least squares solution $\hat{x}$ and

$$\hat{x} = (A^TA)^{-1}A^Tb$$

ex 13

Find the least squares solution to the inconsistent system $Ax = b$ for

$$\mathbf{A} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Note that $A^TA = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$ and it is invertible.

$$(A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \text{ and } A^Tb = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$\implies \hat{x} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Worksheet for Section 13

1. Find a least squares solution of $A\mathbf{x} = \mathbf{b}$. That is, solve for $\hat{\mathbf{x}}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

Homework for Section 13

1. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

   Find a least squares solution of $Ax = b$ by

   (a) constructing the normal equations for $\hat{x}$ and

   (b) solving for $\hat{x}$

14 Inner Product Spaces

Definition 8

An *inner product* on a vector space V, is a function that assigns to each pair of vectors $u, v \in V$ a real number $\langle u, v \rangle$ and satisfies the following axioms:

1. $\langle u, v \rangle = \langle v, u \rangle$
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. $\langle cu, v \rangle = c\langle u, v \rangle$, c a constant
4. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \iff u = 0$

A vector space with an inner product is called an *inner product space*

Note that the length of vector v, or norm, is

$$\|v\| = \sqrt{\langle v, v \rangle} \text{ or } \|v\|^2 = \langle v, v \rangle$$

Theorem 24

CAUCHY-SCHWARZ INEQUALITY

$$\forall u, v \in V$$

$$| \langle u, v \rangle | \leq \|u\| \, \|v\|$$

Theorem 25

THE TRIANGLE INEQUALITY

$$\forall u, v \in V$$

$$\|u+v\| \leq \|u\| + \|v\|$$

proof:

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle = \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle \\
&\leq \|u\|^2 + 2\mid \langle u,v\rangle \mid + \|v\|^2 \\
&\leq \|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2 \\
&\text{(Cauchy-Schwarz Inequality)} \\
&= (\|u\| + \|v\|)^2
\end{aligned}$$

$\blacksquare$

Worksheet for Section 14

1. For vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ both in $\mathbb{R}^2$: let $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 5u_2v_2$.
   Show that defines an inner product.

Homework for Section 14

none

15 Diagonalization of Symmetric Matrices

A symmetric matrix is a matrix A such that $A = A^T$. Obviously then, A needs to be square.

The main diagonal entries are unimportant.

The other entries appear in pairs across the main diagonal.

Can we diagonalize the following?

$$\mathbf{A} = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

Also, is A symmetric?

Given that the characteristic equation is (in factored form, you're welcome...)
$-(\lambda - 8)(\lambda - 6)(\lambda - 3)$
$\Longrightarrow$ the eigenvalues are 8,6 and 3. Now find the corresponding eigenvectors.

For $\lambda = 8$, consider $(A - 8I)x = 0$, row reduce and solve and get

$$\mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 6$, consider $(A - 6I)x = 0$, row reduce and solve and get

$$\mathbf{v_2} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

For $\lambda = 3$, consider $(A - 3I)x = 0$, row reduce and solve and get

$$\mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

so, $A = PDP^{-1}$ with

$$\mathbf{P} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Theorem 26

If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

You can check this with the last example.

proof:

Let $\lambda_1 \neq \lambda_2$ and v_1 be an eigenvector of λ_1 and v_2 be an eigenvector of λ_2

we need to show that $v_1 \bullet v_2 = 0$

$$\begin{gathered}
\lambda_1 v_1 \bullet v_2 = \lambda_1 v_1^T v_2 = (\lambda_1 v_1)^T v_2 \\
= (Av_1)^T v_2 = v_1^T A^T v_2 = v_1^T A v_2 = v_1^T \lambda_2 v_2 \\
= v_1^T v_2 \lambda_2 = v_1 \bullet v_2 \lambda_2 = \lambda_2 v_1 \bullet v_2
\end{gathered}$$

so, $\lambda_1 v_1 \bullet v_2 = \lambda_2 v_1 \bullet v_2$

$$\implies (\lambda_1 - \lambda_2) v_1 \bullet v_2 = 0$$

but since $\lambda_1 \neq \lambda_2 \implies v_1 \bullet v_2 = 0$
■

Definition 9

An $n \times n$ matrix A is orthogonally diagonalizable if $A = PDP^T$. In other words, $P^{-1} = P^T$

Theorem 27

An $n \times n$ matrix A is orthogonally diagonalizable $\iff$ A is symmetric.

The only difference here is that we need to normalize the eigenvectors.

The set of all eigenvalues of A is called the *spectrum* of A.

SPECTRAL THEOREM FOR SYMMETRIC MATRICES

An $n \times n$ symmetric matrix A has the following properties:

1. A has n real eigenvalues, counting multiplicities.
2. The dimension of the eigenspace for each eigenvalue is the multiplicity of λ as a root of the characteristic equation.
3. The eigenspaces are mutually orthogonal.
4. A is orthogonally diagonalizable.

Worksheet for Section 15

1. The matrix A has eigenvalues $\lambda = 5, 2, -2$. Orthogonally diagonalize A. That is, find the orthogonal matrix P and the diagonal matrix D.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Homework for Section 15

1. Let

$$\mathbf{A} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}, \ \mathbf{v_1} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Verify that v_1 and v_2 are eigenvectors of A, then orthogonally diagonalize A.