MATH 150 **Exam 2 Notes** **Fall**

1 The Limit of a Function

Definition

We write:

$$\lim_{x \to a} f(x) = L$$

We say:

the limit of $f(x)$ as x approaches a is L

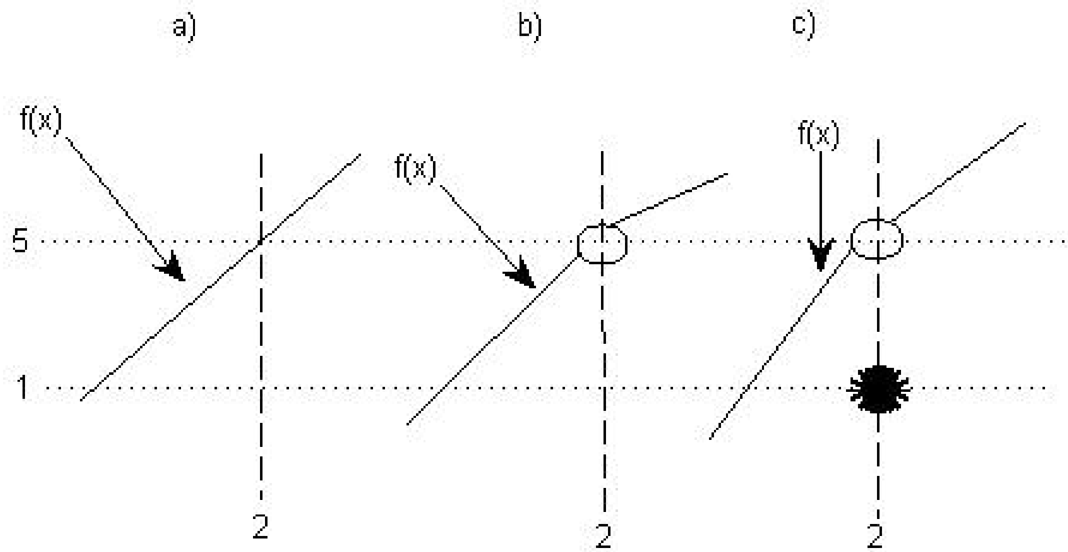
We mean:

we can make the values of $f(x)$ as close to L as we want by choosing values of x sufficiently close to **BUT NOT** equal to a

In fact, $f(x)$ doesn't even need to be defined at a!

ex 1

For each f(x), what is the
lim x->2 f(x) ?



ex 2 Can you guess

$$\lim_{x \to 1} \frac{x-1}{x^2-1}$$

Again, we can use approximations

$x < 1$	$f(x)$
.5	.666667
.99	.502513
.9999	.500025

$x > 1$	$f(x)$
1.5	.4
1.01	.497512
1.0001	.499975

Looks like the limit is .5

What if I changed the function slightly?

$$g(x) = \begin{cases} \frac{x-1}{x^2-1}, & for\ x \neq 1 \\ 10, & for\ x = 1 \end{cases}$$

What is

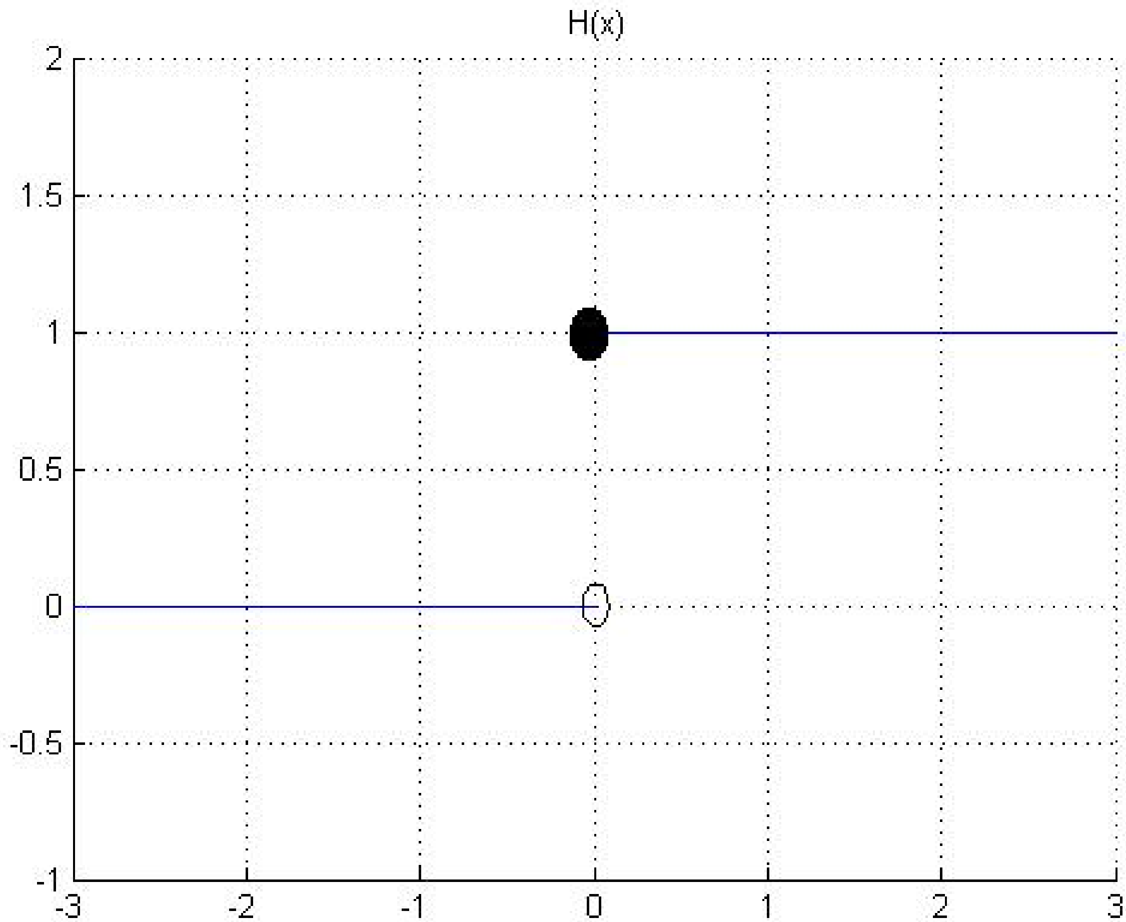
$$\lim_{x \to 1} g(x)$$

ex 3 Let

$$H(x) = \begin{cases} 1, & if\ x \geq 0 \\ 0, & if\ x < 0 \end{cases}$$

Find

$$\lim_{x \to 0} H(x)$$



Note that:

$$\lim_{x \to 0^-} H(x) \;=\; 0 \qquad (from\ the\ left)$$
$$\lim_{x \to 0^+} H(x) \;=\; 1 \qquad (from\ the\ right)$$

Therefore

$$\lim_{x \to 0} H(x) \;=\; DNE\ (does\ not\ exist)$$

Definition

We write:

$$\lim_{x \to a^-} f(x) = L$$

We say:

the limit of $f(x)$ as x approaches a from the left is L

We mean:

we can make the values of $f(x)$ as close to L as we want by choosing values of x sufficiently close to **BUT LESS THAN** a

Also

We write:

$$\lim_{x \to a^+} f(x) = L$$

We say:

the limit of $f(x)$ as x approaches a from the right is L

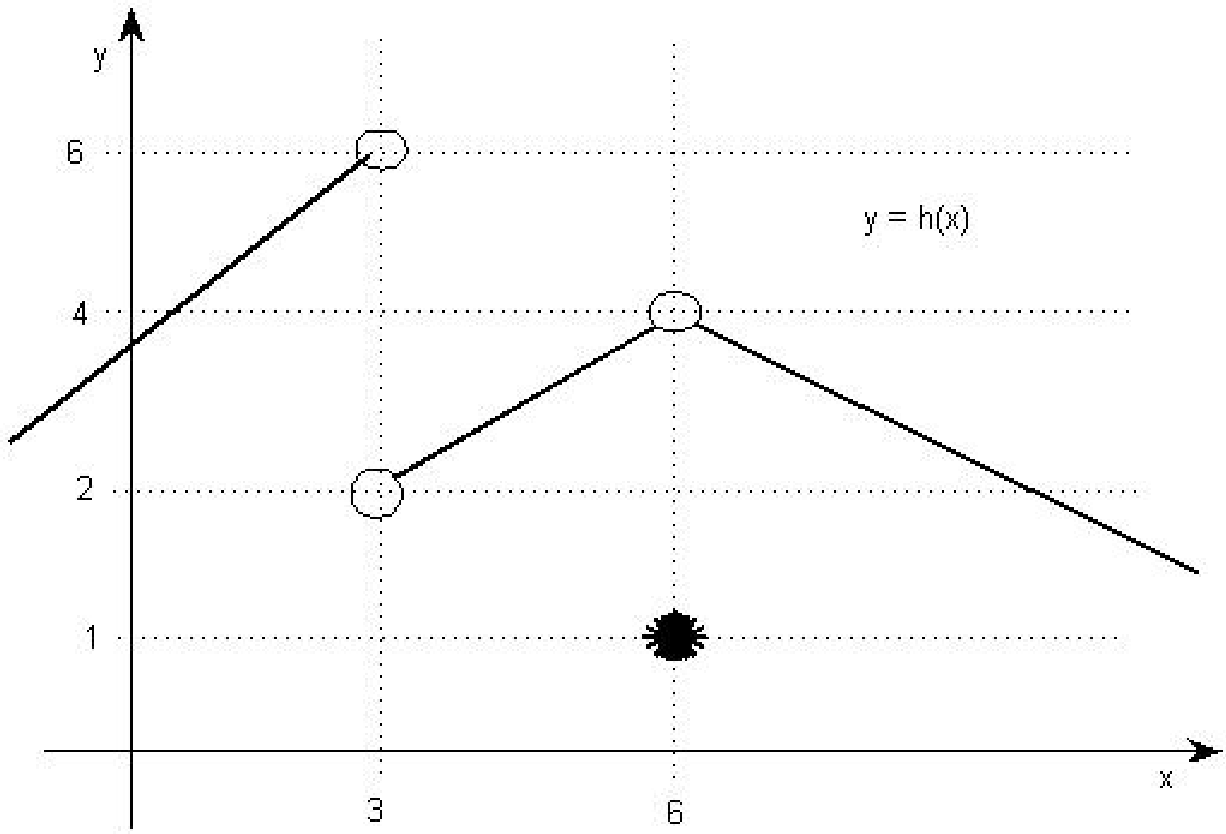
We mean:

we can make the values of $f(x)$ as close to L as we want by choosing values of x sufficiently close to **BUT GREATER THAN** a

Thus

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L \ AND \ \lim_{x \to a^+} f(x) = L$$

ex 4 Find the following, if they exist:



$$\lim_{x \to 3^+} h(x) = \qquad \lim_{x \to 3^-} h(x) = \qquad \lim_{x \to 3} h(x) =$$

$$\lim_{x \to 6^+} h(x) = \qquad \lim_{x \to 6^-} h(x) = \qquad \lim_{x \to 6} h(x) =$$

how about $h(6) = ?$

ex 5 Find

$$\lim_{x \to 0} \frac{1}{x^2}$$

Plug in some small values of x. What do you get? Do these values approach a single number?

NO, thus the limit does not exist, that is, DNE

We say

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

This does **NOT** mean that ∞ is a number or that the limit exists, it is simply a description as to how the limit does not exist.

Thus

We write:

$$\lim_{x \to a} f(x) = \infty$$

We say:

the limit of $f(x)$ as x approaches a is infinity

We mean:

we can make the values of $f(x)$ as large as we want by choosing values of x sufficiently close to **BUT NOT EQUAL** to a

Also

We write:

$$\lim_{x\to a}\ f(x)\ =\ -\infty$$

We say:

the limit of $f(x)$ as x approaches a is negative infinity

We mean:

we can make the values of $f(x)$ as large and negative as we want by choosing values of x sufficiently close to **BUT NOT EQUAL** to a

Note that f(x) need not even be defined at a

Definition

The line $x = a$ is called a vertical asymptote of $y = f(x)$ if at least one of the following is true:

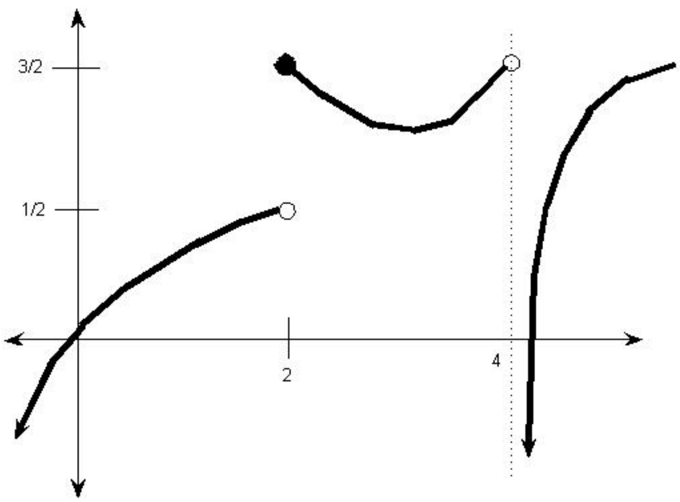
$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a} f(x) = -\infty$$

ex 6 Find

$$\lim_{x\to 0^+}\ ln\ x$$

Worksheet for Section 1



1. For the given function, state the value of each quantity, if it exists. If it does not exist, explain why.

   (a) $\lim_{x\to 2^-}\ g(x)$

   (b) $\lim_{x\to 2^+}\ g(x)$

   (c) $\lim_{x\to 2}\ g(x)$

   (d) $g(2)$

   (e) $\lim_{x\to 4^-}\ g(x)$

   (f) $\lim_{x\to 4^+}\ g(x)$

2. Determine the infinite limit $\lim_{x\to 5^+}\ ln(x-5)$

3. Use a table of values to estimate the value of $\lim_{x\to 0}\ \frac{9^x-5^x}{x}$

Homework for Section 1

1. Sketch a graph of an example of a function f that satisfies all of the given conditions

   (a) $\lim_{x\to 3^+} f(x) = 4, \quad \lim_{x\to 3^-} f(x) = 2, \quad \lim_{x\to -2} f(x) = 2, \quad f(3) = 3, \quad f(-2) = 1$

   (b) $\lim_{x\to 0^-} f(x) = 1, \quad \lim_{x\to 0^+} f(x) = -1, \quad \lim_{x\to 2^-} f(x) = 0, \quad \lim_{x\to 2^+} f(x) = 1, \quad f(2) = 1, \quad f(0)\ is\ undefined$

2. Use a table to estimate the value of

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$$

3. Find the following limits if they exist:

   (a)

$$\lim_{x\to 5^+} \frac{6}{x-5}$$

   (b)

$$\lim_{x\to 1} \frac{2-x}{(x-1)^2}$$

4. In the theory of relativity the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

   where m_0 is the rest mass and c is the speed of light. What happens as $v \longrightarrow c^-$?

2 The Limit Laws

Suppose c is a constant and $\lim_{x\to a}\ f(x)$ and $\lim_{x\to a}\ g(x)$ exists. Then:

1. $\lim_{x\to a}\ [f(x)+g(x)] \ = \ \lim_{x\to a}\ f(x)\ +\ \lim_{x\to a}\ g(x)$
2. $\lim_{x\to a}\ [f(x)-g(x)] \ = \ \lim_{x\to a}\ f(x)\ -\ \lim_{x\to a}\ g(x)$
3. $\lim_{x\to a}\ [c\cdot f(x)] \ = \ c\cdot\lim_{x\to a}\ f(x)$
4. $\lim_{x\to a}\ [f(x)g(x)] \ = \ \lim_{x\to a}\ f(x)\ \lim_{x\to a}\ g(x)$
5. $\lim_{x\to a}\ \frac{f(x)}{g(x)} \ = \ \frac{\lim_{x\to a}\ f(x)}{\lim_{x\to a}\ g(x)}$ *provided* $\lim_{x\to a}\ g(x)\ \neq\ 0$
6. $\lim_{x\to a}\ [f(x)]^n \ = \ [\lim_{x\to a}\ f(x)]^n$ *provided* $n\ \in\ \mathbb{Z}^+$
7. $\lim_{x\to a}\ c\ =\ c$
8. $\lim_{x\to a}\ x\ =\ a$
9. $\lim_{x\to a}\ x^n\ =\ a^n$
10. $\lim_{x\to a}\ \sqrt[n]{x}\ =\ \sqrt[n]{a}$ *provided* $n\ \in\ \mathbb{Z}^+$
11. $\lim_{x\to a}\ \sqrt[n]{f(x)}\ =\ \sqrt[n]{\lim_{x\to a}\ f(x)}$ *provided* $n\ \in\ \mathbb{Z}^+$

These Limit Laws enable us to evaluate limits.

ex 7 Find $\lim_{x\to 5}\ (2x^2-3x+4)$ by justifying each step.

$\lim_{x\to 5}\ (2x^2-3x+4)\ =\ \lim_{x\to 5}\ 2x^2\ -\ \lim_{x\to 5} 3x\ +\ \lim_{x\to 5}\ 4$
(Rule 1 and Rule 2)

$=\ 2\cdot\lim_{x\to 5}\ x^2\ -\ 3\cdot\lim_{x\to 5}\ x\ +\ \lim_{x\to 5}\ 4$
(Rule 3)

$=\ 2(25)\ -\ 3(5)+4$ (Rule 7, Rule 8 and Rule 9)

$$= 39 \quad \text{(Rules from Grade School)}$$

As you can clearly see this is a tedious and boring process. We will not be going about it this way...

Proceed as follows. Given a limit, without justification of course, first try the ***Direct Substitution Property***, which says

If f is a polynomial OR a rational function AND a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

In other words, you can just plug it in. Provided it is in the domain of course!

ex 8 Find

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

Does Direct Substitution work here? Why not? Now what?

If Direct Substitution doesn't work then try factoring or simplifying or rationalizing. Which of those will help here?

Since

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} \implies \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1} =$$

$$\lim_{x\to 1} x+1 = 2$$

BE CAREFUL!! Does

$$\frac{x^2-1}{x-1} = x+1 \text{ ?} \quad NO!$$

Then why is the previous method still correct??

ex 9 Find

$$\lim_{h\to 0} \frac{(4+h)^2 - 16}{h}$$

Again, Direct Substitution will not work. Why? What will work?

$$\lim_{h\to 0} \frac{(4+h)^2 - 16}{h} \;=\; \lim_{h\to 0} \frac{16 + 8h + h^2 - 16}{h} \;=\; \lim_{h\to 0} \frac{8h + h^2}{h} \;=\; \lim_{h\to 0} 8{+}h \;=\; 8$$

Recall that

$$\lim_{x\to a} f(x) \;=\; L \iff \lim_{x\to a^+} f(x) \;=\; L \;\; AND \;\; \lim_{x\to a^-} f(x) \;=\; L$$

ex 10 Show

$$\lim_{x\to 0} \mid x \mid \;=\; 0$$

Recall that:

$$\mid x \mid = \begin{cases} x, & x \geq 0 \\ -x, & x \;<\; 0 \end{cases}$$

If $x \geq 0 \implies \lim_{x\to 0^+} \mid x \mid \;=\; \lim_{x\to 0^+} x \;=\; 0$
If $x < 0 \implies \lim_{x\to 0^-} \mid x \mid \;=\; \lim_{x\to 0^-} (-x) \;=\; 0$
$\implies \lim_{x\to 0} \mid x \mid \;=\; 0$

ex 11 Show

$$\lim_{x\to 0} \frac{\mid x \mid}{x} \quad DNE$$

Again
If $x \geq 0 \implies \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$
If $x < 0 \implies \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{(-x)}{x} = -1$
$\implies \lim_{x\to 0} \frac{|x|}{x} = DNE$

ex 12 Let

$$f(x) = \begin{cases} \sqrt{x-6}, & x > 6 \\ 12 - 2x, & x < 6 \end{cases}$$

does $\lim_{x\to 6} f(x)$ exist? Use the same method as the last two examples.

Theorem

If $f(x) \leq g(x)$ when x is near a and the limits of **BOTH** f and g exist as $x \longrightarrow a$ then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

This essentially provides us with a very useful theorem called....

Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

I will provide a beautiful illustration of this in class.

ex 13 If

$$1 \ \leq \ f(x) \ \leq \ x^2 + 2x + 2$$

for all x, find

$$\lim_{x \to -1} \ f(x)$$

$\lim_{x \to -1} \ 1 \ = \ 1$ and
$\lim_{x \to -1} \ x^2 + 2x + 2 \ = \ 1$ so by the Squeeze Theorem
$\lim_{x \to -1} \ f(x) \ = \ 1$ as well.

Worksheet for Section 2

1. Determine the limit if it exists $\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4}$

2. Find $\lim_{x\to 2} \frac{|x-2|}{x-2}$ if it exists. If it does not exist, explain why.

3. Show by means of an example that $\lim_{x\to a} [f(x)g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

Homework for Section 2

1. Is the following a true statement?

$$\frac{x^2+x-6}{x-2} \;=\; x+3$$

2. Why is this statement true then?

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2} = \lim_{x\to 2}\; x+3$$

3. Find the following limits if they exist:

   (a) $\lim_{x\to 2}\ \frac{x^2+x-6}{x-2}$

   (b) $\lim_{x\to 2}\ \frac{x^2+x+6}{x-2}$

   (c) $\lim_{x\to -3}\ \frac{x^2-9}{2x^2+7x+3}$

   (d) $\lim_{h\to 0}\ \frac{(4+h)^2-16}{h}$

   (e) $\lim_{x\to 1}\ \frac{x^3-1}{x^2-1}$

   (f) $\lim_{t\to 9}\ \frac{9-t}{3-\sqrt{t}}$

   (g) $\lim_{x\to 7}\ \frac{\sqrt{x+2}-3}{x-7}$

   (h) $\lim_{x\to -4}\ \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

   (i) $\lim_{x\to -4}\ \mid x+4 \mid$

   (j) $\lim_{x\to 4}\ \frac{|x-4|}{x-4}$

4. Let $f(x) \;=\begin{cases} 4-x^2 & if\ \ x\le 2 \\ x-1 & if\ \ x>2 \end{cases}$

   (a) Find $\lim_{x\to 2^-}\ f(x)$

   (b) Find $\lim_{x\to 2^+}\ f(x)$

   (c) Find $\lim_{x\to 2}\ f(x)$

5. If $f(x) \;=\begin{cases} x^2 & if\ x\ is\ rational \\ 0 & if\ x\ is\ irrational \end{cases}$ prove $\lim_{x\to 0}\ f(x) \;=\; 0$

6. Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find a and the limit.

3 The Precise Definition of a Limit

Phrases like "as close as you like" are sometimes inadequate.

Consider

$$f(x) = \begin{cases} 2x-1 & if \;\; x \neq 3 \\ 6 & if \;\; x = 3 \end{cases}$$

Intuitively it should be clear that when x is close to 3, but $x \neq 3$, then $f(x)$ is close to 5.

But, how close to 3 must x be so that $f(x)$ differs from 5 by less than 0.1?

Now

$\mid x - 3 \mid$ represents the distance from x to 3

$\mid f(x) - 5 \mid$ represents the distance from $f(x)$ to 5

So we need to find a number, which we will call delta δ, such that

$$If \quad 0 <\mid x-3 \mid< \delta \quad then \quad \mid f(x) - 5 \mid< 0.1$$

Notice that if $\delta = \frac{0.1}{2} \;=\; .05$, then

$$\mid f(x) - 5 \mid \;=\; \mid 2x - 1 - 5 \mid \;=\; \mid 2x - 6 \mid \;=\; 2 \mid x - 3 \mid \;<\; 0.1$$

ONLY if

$$0 \;<\; \mid x - 3 \mid \;<\; .05 \;=\; \delta$$

What if I change 0.1 to .01, will the same δ work? No.

Using the same method as above, we would need δ to be .005

You can see that this game can go on for quite a while. The idea is to be able to bring the difference between $f(x)$ and 5 below ANY positive number.

Clearly an actual value will not work since for any value of δ someone can just choose a much smaller value. From the example though we can see that the value of δ depended on how close I wanted $f(x)$ and 5 to be.

For 0.1 we needed $\delta = .05$
For .01 we needed $\delta = .005$, etc...

The proximity of $f(x)$ and 5 also gets a value, it is called epsilon ϵ

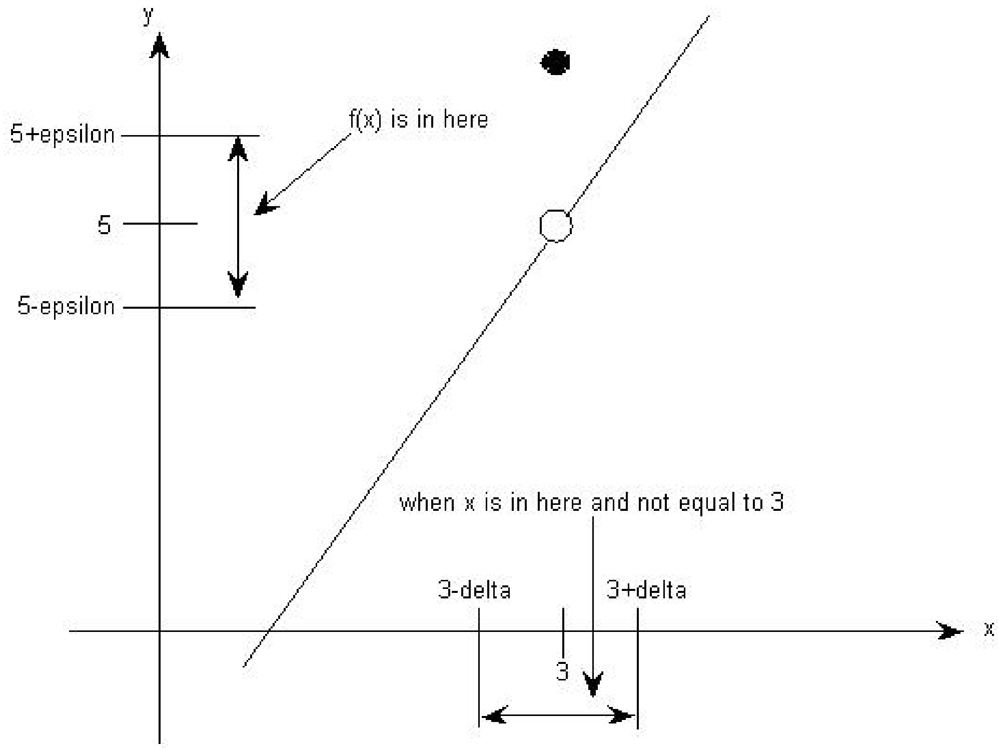
So here is how the game is played...

For this example, let ϵ be any arbitrary positive number, 0.1, .001 or whatever, then

$$If \;\; 0 < \; |\, x - 3 \,| \; < \; \delta \; = \; ? \; \Longrightarrow \; |\, f(x) - 5 \,| \; < \; \epsilon$$

$$If \;\; ? \; = \; \frac{\epsilon}{2} \;\; then\ this\ will\ ALWAYS\ work\ !$$

Graphically



If we take values of x $(x \neq 3)$ to lie in the interval $(3-\delta, 3+\delta)$ we can then make values of $f(x)$ lie in the interval $(5-\epsilon, 5+\epsilon)$

Definition

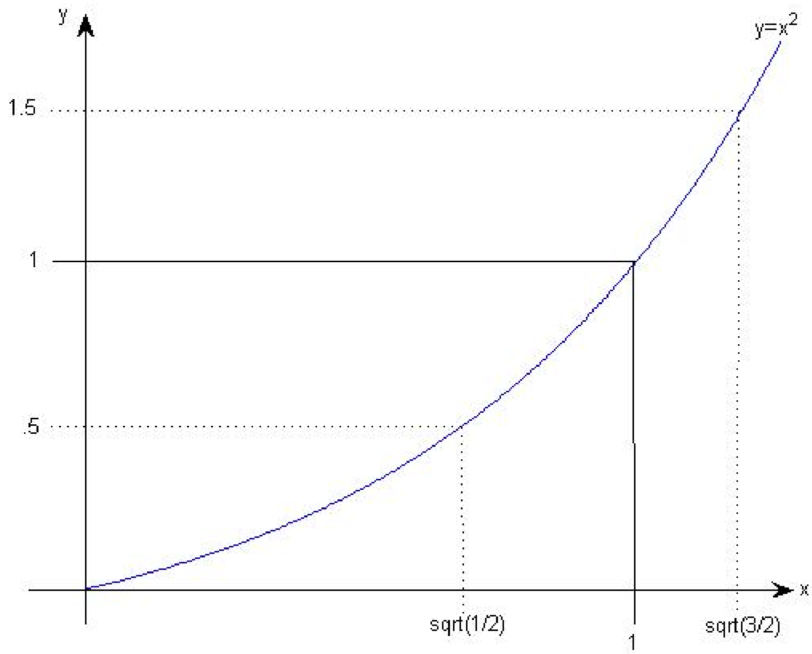
Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) \;=\; L$$

IF

$$\forall\, \epsilon \;>\; 0\,,\; \exists\, \delta \;>\; 0 \;\; \textit{such that} \;\; |\, x - a \,| \;<\; \delta \;\Longrightarrow\; |\, f(x) - L \,| \;<\; \epsilon$$

ex 14 given $y = x^2$, find a δ such that $|\, x - 1 \,| < \delta \implies |\, x^2 - 1 \,| < \frac{1}{2}$



For this example, the equivalent limit would read..

$$\lim_{x \to 1} x^2 = 1$$

on the left hand side we need $| x - 1 | < | \frac{1}{\sqrt{2}} - 1 | \approx .292$

on the right hand side we need $| x - 1 | < | \sqrt{\frac{3}{2}} - 1 | \approx .224$

So we must choose what for δ? .224 but why?

Always the more restrictive of the two. What works on one side must work on the other as well.

ex 15 Prove $\lim_{x \to 3} (4x - 5) = 7$

The idea is to use the definition and work backwards. The definition says:

$$\forall \epsilon > 0 , \exists \delta > 0 \; such\, that \; | x - a | < \delta \implies | f(x) - L | < \epsilon$$

Which means we need to end up with $|\, f(x) - L \,| \;\; < \; \epsilon$ so we will do our SCRATCH WORK

$$We\, need \;\; |\, f(x){-}L \,| \;\; < \; \epsilon \;\; \Longrightarrow \;\; |\,(4x{-}5){-}7\,| \;\; < \; \epsilon \;\; \Longrightarrow \;\; |\,4x{-}12\,| \;\; < \; \epsilon$$
$$\Longrightarrow$$
$$|\,4\,||\,x-3\,| \;\; < \; \epsilon \;\; \Longrightarrow \;\; 4\,|\,x-3\,| \;\; < \; \epsilon$$

Now what do we have control over? Why did I stop when I got $|\,x-3\,|$ as a factor? That represents δ which I can make as small as I want! Remember that you are always given ϵ and you get to pick the δ

That is, given $|\,x-3\,| \;\; < \; \delta \;\; \Longrightarrow \;\; \ldots$

What might work for δ?

Why does $\frac{\epsilon}{4}$ work?

Now formulate the proof

$$Given \;\; \epsilon > 0, \;\; choose \; \delta = \frac{\epsilon}{4} \; (or\ anything\ smaller).$$
$$Then\ if \;\; |\,x-3\,| \;\; < \; \delta \;\; we\ have\ that$$
$$|\,(4x-5)-7\,| \;\; = \;\; |\,4x-12\,| \;\; = \;\; |\,4\,||\,x-3\,| \;\; < \; 4\cdot\delta \; = \; 4\cdot\frac{\epsilon}{4} \; = \; \epsilon$$
$$Thus, \;\; |\,x-3\,| \;\; < \; \delta \;\; \Longrightarrow \;\; |\,(4x-5)-7\,| \;\; < \; \epsilon \qquad \blacksquare$$

Of course this works perfectly with actual values as well. Let's try one. Say I want you to get within .001 of $f(x) = 4x - 5$

Since $\lim_{x \to 3} \ (4x - 5) = 7$, I want to see how far away from 3 I need to be so that I am within the interval $(6.999, 7.001)$

Now our proof claims that $\delta \ = \ \frac{\epsilon}{4}$ or anything smaller of course so what should work? Well, $\frac{\epsilon}{4} \ = \ \frac{.001}{4} \ = \ .00025$

This means if we are in the interval $(2.99975, 3.00025)$ on the x-axis then we should be in $(6.999, 7.001)$ on the y-axis

Let's see. What is $f(2.9998)$? It's 6.9992. That is in our prescribed interval so I guess that works.
What if we are just outside the interval at say $f(2.99970)$? Well $f(2.99970)$ is 6.9988 which is indeed out of our y-axis window.

I would say that is moderately neato wouldn't you? Be honest, I know you would.

ex 16 Prove $\lim_{x \to 1} \ (2x + 3) = 5$

SCRATCH WORK

$$We\ need \quad |\, f(x) - L \,| \ < \epsilon \ \Longrightarrow \quad |\, (2x + 3) - 5 \,| \ < \epsilon \ \Longrightarrow$$
$$|\, 2x - 2 \,| \ < \epsilon \ \Longrightarrow$$
$$|\, 2 \,||\, x - 1 \,| \ < \epsilon \ \Longrightarrow \ 2 \,|\, x - 1 \,| \ < \epsilon$$

What works for δ now?

Now formulate the proof

$$Given\ \epsilon > 0,\ \ choose\ \delta = \frac{\epsilon}{2}\ (or\ anything\ smaller).\ Then\ if\ \ |\, x{-}1\,|\ \ < \delta$$
$$we\ have\ that$$

$$|\,(2x+3)-5\,|\ \ =\ \ |\,2x-2\,|\ \ =\ \ |\,2\,||\,x-1\,|\ \ <\ 2\cdot\delta\ =\ 2\cdot\frac{\epsilon}{2}\ =\ \epsilon$$

$$Thus,\ \ \ |\,x-1\,|\ \ <\ \delta\ \ \Longrightarrow\ \ \ |\,(2x+3)-5\,|\ \ <\ \epsilon\qquad\blacksquare$$

Try some actual values with this one and you will see that it works perfectly.

With these past couple of examples the trend seems to be, factor out a constant so we can get the $|\ x - a\ |$ part times the constant, and then divide by the constant to find δ. As long as we are dealing with constants this works fine because since we obtain $c\,|\ x - a\ |$ and we get to make $|\ x - a\ |$ as small as we want. What happens if we get a little extra though? Does this method still work?

ex 17 Prove $\lim_{x\to 3}\ x^2 = 9$

SCRATCH WORK

$$We\ need\ \ \ |\,f(x) - L\,|\ \ <\ \epsilon\ \ \Longrightarrow\ \ \ |\,x^2 - 9\,|\ \ <\ \epsilon$$
$$\Longrightarrow\ \ \ |\,(x+3)(x-3)\,|\ \ <\ \epsilon\ \ \Longrightarrow$$
$$|\,x+3\,||\,x-3\,|\ \ <\ \epsilon$$

And uh oh. What works for δ now?

$$\delta \;=\; \frac{\epsilon}{\mid x+3 \mid}\;????$$

Absolutely not! We need δ to be a constant.

Well what if we could find a constant, C, such that $\mid x+3 \mid < C$. We are only really interested in values of x close to 3 anyway. No one would be silly enough to say get me within the interval $(0,18)$ on the y-axis would they?

If we could find this C, then we could just let $\delta = \frac{\epsilon}{C}$ and repeat the previous procedure so let's see if we can do just that.

Since we are taking a limit as $x \to 3$, we are only concerned with values close to 3. How close? Let's assume within 1 unit of 3. (this 1 unit is arbitrary, you could choose however many units you like)

Keep in mind we need $\mid x+3 \mid < C$, so within 1 unit of 3 means $\mid x-3 \mid < 1$

If we choose 2 units we would start with $\mid x-3 \mid < 2$, etc...

So $\;\mid x-3 \mid \; < 1 \implies 2 < x < 4 \implies 5 < x+3 < 7 \implies \; \mid x+3 \mid < 7$

So it seems that $C = 7$ is a suitable choice.

We must proceed carefully since now we really have 2(two) restrictions:

1. $\mid x+3 \mid < 7$

   AND

2. $\mid x-3 \mid < \frac{\epsilon}{C} = \frac{\epsilon}{7}$

To insure that both of these restrictions are met you must choose δ to be the **smaller** of the 2 (remember ex 46)

That is denoted as $\delta = min\{1, \epsilon/7\}$
Now formulate the proof

$$Given \;\; \epsilon > 0, \;\; choose \; \delta = min\left\{1, \frac{\epsilon}{7}\right\}. \;\; Then \; if \;\; \mid x-3 \mid \; < \; \delta$$

$$we \; have \; that$$

$$\mid x-3 \mid < 1 \; so \; that \;\; \mid x+3 \mid < 7$$

$$Thus, \quad \mid x^2 - 9 \mid \;\; = \;\; \mid x+3 \mid\mid x-3 \mid \;\; < \;\; 7\left(\frac{\epsilon}{7}\right) \;\; = \; \epsilon \qquad \blacksquare$$

You really need the minimum aspect. If you only said $\delta = \frac{\epsilon}{7}$ then try it with $\epsilon = 14$ and you will see that it fails.

Clearly though $min\{1, \frac{14}{7}\}$ is 1 and it would follow.

Worksheet for Section 3

1. Use a graph of $f(x) = 1/x$ to find a number δ such that $\left| \frac{1}{x} - 0.5 \right| < 0.2$ whenever $| x - 2 | < \delta$.

2. Prove, using the δ , ϵ definition of a limit, that $\lim_{x \to -2} (\frac{1}{2}x + 3) = 2$

Homework for Section 3

1. Graph $f(x) = \sqrt{x}$ and use it to find a number δ such that $\mid \sqrt{x} - 2 \mid < 0.4$ whenever $\mid x - 4 \mid < \delta$

2. Prove the following statements using the ϵ, δ definition.

   (a) $\lim_{x \to -3} \ (1 - 4x) = 13$

   (b) $\lim_{x \to 4} \ (7 - 3x) = -5$

   (c) $\lim_{x \to 3} \ \frac{x}{5} = \frac{3}{5}$

   (d) $\lim_{x \to -2} \ (x^2 - 1) = 3$

3. If $f(x) \ = \begin{cases} 0 & if \ x \ is \ rational \\ 1 & if \ x \ is \ irrational \end{cases}$ prove $\lim_{x \to 0} \ f(x)$ does not exist.

4 Continuity

Definition

A function is *continuous at a number a* if

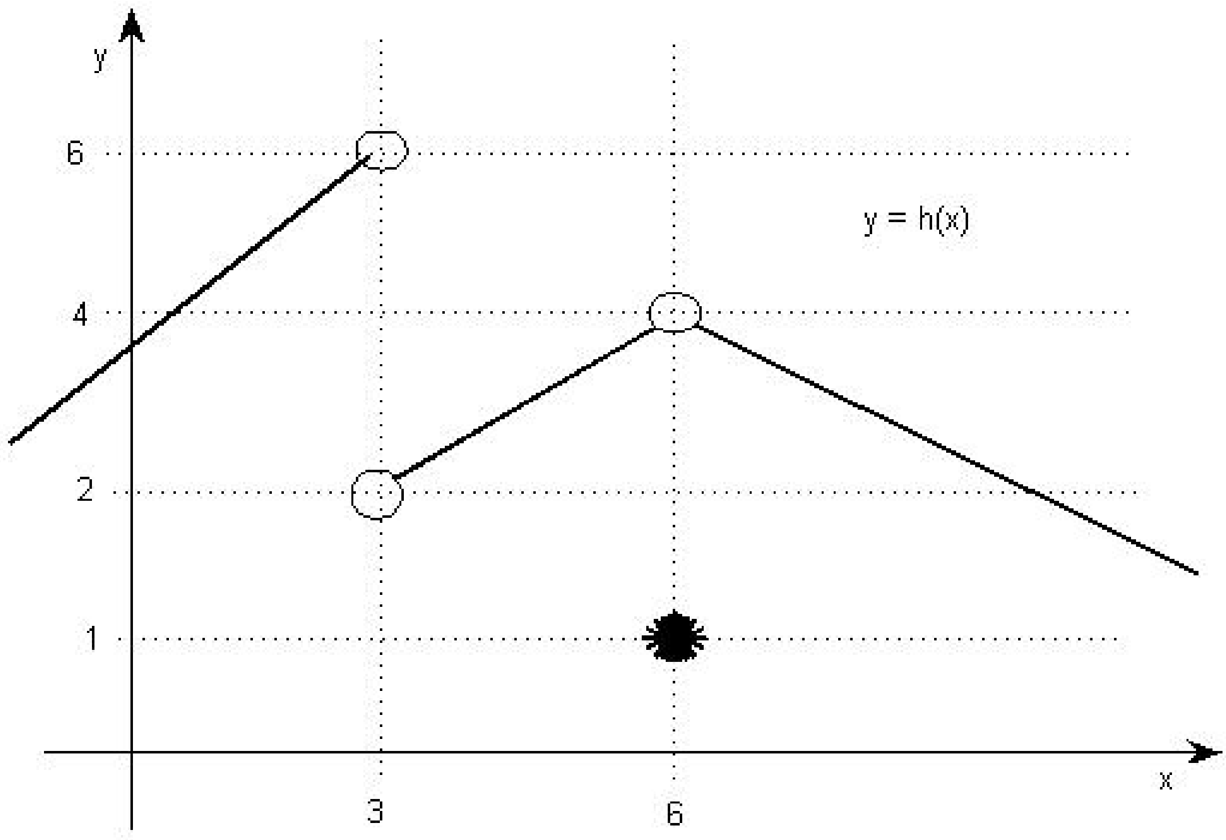
$$\lim_{x \to a} f(x) = f(a)$$

It is absolutely crucial to realize that this definition requires three (3) things:

1. $f(a)$ is defined, that is, a is in the domain of f
2. $\lim_{x \to a} f(x)$ exists, AND
3. $\lim_{x \to a} f(x) = f(a)$, that is, the numbers you get from (1) and (2) are THE SAME

If f is not continuous, it is called *discontinuous*

ex 18 Where is $h(x)$ discontinuous?



at $a = 3$, since both (1) and (2) are violated and at

$a = 6$ since (3) is violated $\qquad h(6) = 1$ and $\lim_{x\to 6} h(x) = 4$ and clearly $1 \neq 4$

ex 19 Where is $f(x)$ discontinuous?

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

at $x = 2$ Why?

ex 20 Where is $f(x)$ discontinuous?

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

still at $x = 2$ Why?

Some books identify different types of discontinuities, such as removable, infinite and jump. I will graph these for you.

Definition

A function f is *continuous from the right* at a if

$$\lim_{x \to a^+} f(x) = f(a)$$

Definition

A function f is *continuous from the left* at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition

A function f is *continuous on an interval* if it is continuous at every number on that interval.

At an endpoint it is understood to be continuous from either the right or left

Theorem

If f and g are continuous at a and c is a constant, then the following are also continuous at a

1. $f + g$
2. $f - g$
3. $c(f)$
4. fg
5. f/g, provided $g \neq 0$

Theorem

Any polynomial is continuous on $\mathbb{R}$

Any Rational Function is continuous on its domain

Theorem

The following functions are continuous at every number *in their domains*:

polynomials trig functions log functions exponential functions
rational functions inverse trig functions root functions

these are called the elementary functions

ex 21 How can we show that $f(x)$ is continuous on $(-\infty, \infty)$ if

$$f(x) = \begin{cases} x^2, & if\ x < 1 \\ \sqrt{x}, & if\ x \geq 1 \end{cases}$$

$f(x)$ is a polynomial on $(-\infty, 1)$ so it is continuous there
$f(x)$ is a root function on $(1, \infty)$ so it is continuous there

we need to make sure that these two parts "meet" at $x = 1$, so lets go through the three possibilities at $x = 1$ since that is the only possible problem

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x} = 1$$
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$$

thus, $\lim_{x \to 1} f(x) = 1$, also, $f(1) = \sqrt{1} = 1 \implies f(x)$ is continuous on $(-\infty, \infty)$

Theorem

If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$ that is,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

- A limit symbol can be moved through a function symbol **IF** the function is continuous and the limit exists

ex 22 Evaluate

$$\lim_{x\to 1} arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$$

Since arcsin is continuous this can be rewritten as

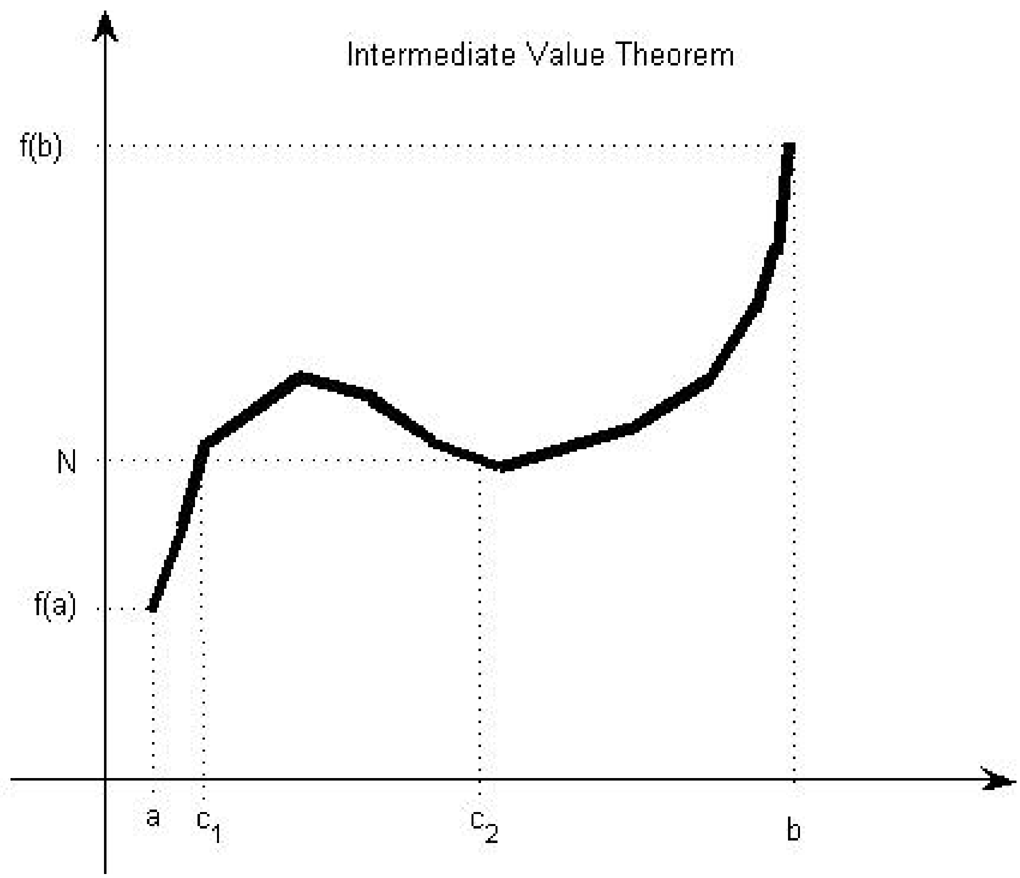
$$arcsin\lim_{x\to 1}\left(\frac{1-\sqrt{x}}{1-x}\right) =$$
$$arcsin\lim_{x\to 1}\left(\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right) =$$
$$arcsin\lim_{x\to 1}\left(\frac{1}{1+\sqrt{x}}\right) =$$
$$arcsin\,\frac{1}{2} = \frac{\pi}{6}$$

Theorem

If g is continuous at a and f is continuous at $g(a)$ then the composite function, $f \circ g$ is continuous.

The Intermediate Value Theorem

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there must exist a number c in (a, b) such that $f(c) = N$



Here there are 2 values where $f(c) = N$, c_1 and c_2

ex 23 How can we show that there is a root of $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2?

So, we are looking for a c between 1 and 2 such that $f(c) = 0$

But, $f(1) = -1 \ < \ 0$ and
$f(2) = 12 \ > \ 0$ and since f is continuous, by the IVT there must be at least one root between 1 and 2. Why?

Worksheet for Section 4

1. Show that $f(x)$ is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} x^2, & x < 1 \\ \sqrt{x}, & x \geq 1 \end{cases}$$

2. Why is the following function discontinuous at $a = 1$?

$$g(x) = \begin{cases} \frac{x^2 - x}{x^2 - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

3. Is there a number that is exactly one more than its cube?

Homework for Section 4

1. Sketch the graph of a function that is continuous everywhere except at $x = 2$

2. Use the definition to show that $f(x) = x^2 + \sqrt{7 - x}$ is continuous at 4.

3. Explain why the following functions are discontinuous at the given number a

   (a) $f(x) = ln \ \mid x - 2 \mid \qquad a = 2$

   (b) $f(x) \ = \begin{cases} \frac{1}{x-1} & if \ \ x \neq 1 \\ 2 & if \ \ x = 1 \end{cases} \qquad a = 1$

   (c) $f(x) \ = \begin{cases} e^x & if \ \ x < 0 \\ x^2 & if \ \ x \geq 0 \end{cases} \qquad a = 0$

4. Find the numbers where the following functions are discontinuous

   (a) $f(x) \ = \begin{cases} 1 + x^2 & if \ \ x \leq 0 \\ 2 - x & if \ \ 0 < x \leq 2 \\ (x - 2)^2 & if \ \ x > 2 \end{cases}$

   (b) $f(x) \ = \begin{cases} x + 2 & if \ \ x < 0 \\ e^x & if \ \ 0 \leq x \leq 1 \\ 2 - x & if \ \ x > 1 \end{cases}$

5. For what constant c is the following function continuous on $(-\infty, \infty)$.

   $f(x) \ = \begin{cases} cx + 1 & if \ \ x \leq 3 \\ cx^2 - 1 & if \ \ x > 3 \end{cases}$

6. If $f(x) = x^3 - x^2 + x$ show there is a number c such that $f(c) = 10$.

7. Use the IVT to show that $f(x) = x^4 + x - 3 = 0$ must have a root in $(1, 2)$

8. For what values is the following function continuous? $f(x) = \begin{cases} 0 & if\ x\ is\ rational \\ 1 & if\ x\ is\ irrational \end{cases}$

5 Limits at Infinity

Consider the following limit:

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1}$$

If you begin to plug in values of x, what happens?

It seems as if the fraction gets closer and closer to 1

Definition

We write:

$$\lim_{x\to\infty}\ f(x)\ =\ L$$

We say:

the limit of $f(x)$ as x approaches infinity is L

We mean:

we can make the values of $f(x)$ as close to L as we want by choosing values of x sufficiently large

Also

We write:

$$\lim_{x\to-\infty}\ f(x)\ =\ L$$

We say:

the limit of $f(x)$ as x approaches negative infinity is L

We mean:

we can make the values of $f(x)$ as close to L as we want by choosing values of x sufficiently large and negative

* Note that ∞ and $-\infty$ do NOT represent actual numbers *

Definition

The line $y = L$ is called a *horizontal asymptote* of $y = f(x)$ if:

$$\lim_{x \to \infty} f(x) = L$$

or

$$\lim_{x \to -\infty} f(x) = L$$

ex 24 Investigate

$$\lim_{x \to \infty} \frac{1}{x}$$

What happens to the function as x gets very large? It looks like the limit is 0.

Theorem

If $r > 0$ is rational then,

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is rational such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

We will use this fact to evaluate limits as x approaches ∞ OR $-\infty$

ex 25 Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

The technique is as follows. Since we are only concerned with large values of x, divide EACH TERM by the highest power in the DENOMINATOR. Then we can utilize the previous theorem.

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

Now, keeping in mind the theorem, what happens to the following as $x \longrightarrow \infty$?

$$\frac{1}{x}, \ \frac{2}{x^2}, \ \frac{4}{x} \ \ and \ \ \frac{1}{x^2}$$

They all go to 0! Thus,

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

ex 26 Find the vertical and horizontal asymptotes of

$$y = \frac{x^3}{x^2 + 3x - 10}$$

Recall that to find the vertical asymptotes simply determine when the denominator is zero

To find the horizontal ones, determine the limit as y approaches both ∞ and $-\infty$

ex 27 Find

$$\lim_{x \to \infty} e^x$$

What happens as x gets very large? So does e^x

ex 28 Evaluate

$$\lim_{x \to -\infty} e^x$$

Don't forget that e^{-x} is equivalent to $\frac{1}{e^x}$ therefore this limit gets very small

ex 29 How about

$$\lim_{x \to \infty} sin\ x$$

What happens to the graph of $y = sin\ x$ as you go farther and farther to the right?

ex 30 Evaluate

$$\lim_{x \to \infty} \frac{x^2 + x - 5}{15 - x}$$

Try this one on your own

Worksheet for Section 5

1. Sketch a graph of a function that satisfies the following conditions:

   (a) $\lim_{x\to -2}\ f(x) = \infty$

   (b) $\lim_{x\to -\infty}\ f(x) = 3$

   (c) $\lim_{x\to \infty}\ f(x) = -3$

2. Find $\lim_{x\to\infty} \frac{x^3-2x+3}{5-2x^2}$

3. Find $\lim_{x\to\infty} tan^{-1}(x^2 - x^4)$

Homework for Section 5

1. Sketch the graph of an example of a function that satisfies the given conditions.

   (a) $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 1$, $\lim_{x\to-\infty} f(x) = 1$

   (b) $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 2} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$

2. Find the following limits if they exist:

   (a)
   $$\lim_{x\to\infty} \frac{3x+5}{x-4}$$

   (b)
   $$\lim_{x\to\infty} \frac{2-3x^2}{5x^2+4x}$$

   (c)
   $$\lim_{x\to-\infty} \frac{x^2+2}{x^3+x^2-1}$$

   (d)
   $$\lim_{u\to\infty} \frac{4u^4+5}{(u^2-2)(2u^2-1)}$$

   (e)
   $$\lim_{x\to\infty} \frac{\sqrt{9x^6-x}}{x^3+1}$$

   (f)
   $$\lim_{x\to\infty} cos\ x$$

   (g)
   $$\lim_{x\to\infty} \sqrt{x}$$

3. Find the horizontal and vertical asymptotes of $y = \frac{x^2+4}{x^2-1}$

6 The Derivative

Recall

Previously we used approximations to determine the slope of the tangent line. That is, given $y = f(x)$ and a point $P(a, f(a))$ we considered a nearby point $Q(x, f(x))$ where $x \neq a$ and found the slope of the secant line PQ, which was

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Now

Definition 1

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Limits are how we formally deal with the concept of getting closer and closer

If you recall the example from before we attempted to find the slope of the tangent line to the curve $y = x^2$ at the point $(1, 1)$. Now we can determine this exactly. The slope is:

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \dots = 2$$

There is an analagous definition that we will also use

If we let $h = x - a \quad \Longrightarrow \quad x = a + h$ and

$$m_{PQ} \;=\; \frac{f(a+h) - f(a)}{h}$$

So

Definition 2

$$m \;=\; \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

***** Definitions 1 and 2 are two different ways to say *the exact same thing* *****

ex 31 Find the EQUATION of the tangent line to $y = 3/x$ at the point $(3, 1)$

Now in order to find the equation of ANY line you need either two points OR one point and the slope.

How fortunate for us that we now have a formula to determine the slope.

Let's use definition 2:
Since $f(x) = 3/x$ we have

$$m \;=\; \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \;=\; \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h} \;=\; \lim_{h \to 0} \frac{\frac{3}{3+h} - \frac{3+h}{3+h}}{h} \;=\; \lim_{h \to 0} \frac{\frac{3-(3+h)}{3+h}}{h} \;=\; \lim_{h \to 0} \frac{\frac{-h}{3+h}}{h} \;=\; \lim_{h \to 0} \frac{-h}{h(3+h)} \;=$$

$$\lim_{h \to 0} \; -\frac{1}{3+h} \;=\; -\frac{1}{3}$$

So the EQUATION is $y - 1 \;=\; -1/3(x-3)$

Note that I have used point-slope form. You may use y-intercept if you insist but it will require some more work on your part to determine b

Velocities

In general the equation of motion is given by $s = f(t)$ where s is the displacement of the object and t is time. This function, $s = f(t)$, is called the position function. Now the average velocity is simply the change in distance, or displacement, divided by the change in time. That is:

$$average\ velocity \;=\; \frac{\Delta displacement}{\Delta time} \;=\; \frac{f(a+h)-f(a)}{h}$$

with a start time of $t = a$ and an end time of $t = a + h$

or you can think of h as the change in time if you like

How would I go from average velocity (two points) to instantaneous velocity (one point)?

$$instantaneous\ velocity \;=\; v(a) \;=\; \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Look familiar??

For $y = f(x)$, if x changes from x_1 to x_2, let $\Delta x = x_2 - x_1$

Then the ***average rate of change***, also called the ***difference quotient***, is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

OR

$$\frac{f(a+h) - f(a)}{h}$$

And the ***instantaneous rate of change*** is:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

OR

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The instantaneous velocity is so important that it gets a special name. It is called **The Derivative**

Definition

The derivative of a function f at the number a, denoted $f'(a)$, is:

$$f'(a) \ = \ \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

Equivalently, of course,

$$f'(a) \ = \ \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

ex 32 Find the derivative of $2x^2 - 5$ at the number a

So

$$f'(a) \ = \ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \ = \ \lim_{x \to a} \frac{2x^2 - 5 - (2a^2 - 5)}{x - a} \ = \ \lim_{x \to a} \frac{2x^2 - 2a^2}{x - a}$$

$$= \ \lim_{x \to a} 2\,\frac{x^2 - a^2}{x - a} \ = \ \lim_{x \to a} 2\,\frac{(x+a)(x-a)}{x - a} \ = \ 2 \lim_{x \to a} x+a \ = \ 2(2a) \ = \ 4a$$

ex 33 Find the equation of the tangent line to the graph $y = 2x^2 - 5$ at the point $(2, 3)$

Since we have already determined the slope at any point in the previous example, $m = 4a$, the slope when $a = 2$ is 8.

Thus the equation is $y - 3 \ = \ 8(x - 2)$

ex 34 If the position of a particle is $s = f(t) = \frac{1}{1+t}$ where t is in seconds and s is in meters, find the velocity and speed after 2 seconds

Let's use the other definition this time. $f(t) = \frac{1}{1+t}$, so

$$
\begin{aligned}
v(2) \;=\; f'(2) \;&=\; \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \;=\; \lim_{h\to 0} \frac{\frac{1}{3+h}-\frac{1}{3}}{h} \\
&= \lim_{h\to 0} \frac{\frac{3}{3(3+h)}-\frac{3+h}{3(3+h)}}{h} \;=\; \lim_{h\to 0} \frac{\frac{-h}{3(3+h)}}{h} \;=\; \lim_{h\to 0} \frac{-h}{3(3+h)h} \\
&= \lim_{h\to 0} \frac{-1}{3(3+h)} \;=\; -\frac{1}{9}
\end{aligned}
$$

Thus the velocity is $-1/9$ and the speed is $1/9$. Why?

Worksheet for Section 6

1. If a ball is thrown into the air with a velocity of 40 ft/s, its height, in feet, after t seconds is given by $y = 40t - 16t^2$. Find the average velocity from $t = 1$ to $t = 2$ as well as the velocity when $t = 2$.

2. Find $f'(a)$ if

$$f(x) = \frac{1}{\sqrt{x+2}}$$

3. Given that

$$\lim_{h \to 0} \frac{cos(\pi + h) + 1}{h}$$

represents $f'(a)$, what is the function f and the number a?

Homework for Section 6

1. Find the slope of the tangent line at the given point for the following functions:

   (a) $f(x) = x^2 + 2x$ at $(-3, 3)$

   (b) $f(x) = \sqrt{2x+1}$ at $(4, 3)$

   (c) $f(x) = \frac{1}{x+1}$ at $(1, 1/2)$

2. A ball is thrown into the air so that its height in feet after t seconds is $s(t) = t^2 - 8t + 18$

   (a) What is the average velocity over the following intervals?

      i. $[3, 4]$

      ii. $[3.5, 4]$

      iii. $[4, 5]$

      iv. $[4, 4.5]$

   (b) What is the instantaneous velocity at $t = 4$?

3. A different ball is thrown into the air so that its height in feet after t seconds is $h(t) = 58t - 0.83t^2$

   (a) What is the velocity after one second?

   (b) When will the ball strike the ground?

   (c) What is the velocity at this time?

4. Sketch a graph of a function f that satisfies all of the given conditions:

   (a) $f(0) = 0,\ f'(0) = 3,\ \ f'(1) = 0,\ \ and\ \ f'(2) = -1$

   (b) $f(0) = 0,\ f'(0) = 3,\ \ f'(1) = 0,\ \ and\ \ f'(2) = 1$

5. Find $f'(a)$ for the following functions:

   (a) $f(x) = x^2 - 2x + 2$

(b) $f(x) = \frac{2x+1}{x+3}$

(c) $f(x) = \sqrt{3x+1}$

6. Each limit represents the derivative for some f at some number a. Find f and a for each.

   (a) $\lim_{h\to 0} \frac{(1+h)^{10}-1}{h}$

   (b) $\lim_{x\to 5} \frac{2^x-32}{x-5}$

   (c) $\lim_{x\to \pi/4} \frac{tan\ x-1}{x-\pi/4}$

7. Find the velocity when $t = 2$ if the displacement is given by $s(t) = t^2 - 6t - 5$.

8. The number of bacteria after t hours is $n = f(t)$.

   (a) What is meant by $f'(5)$?

   (b) Given unlimited space and nutrients, which is larger, $f'(5)$ or $f'(10)$?

   (c) Given limited space and nutrients, would you change your answer? Why?

7 The Derivative as a Function

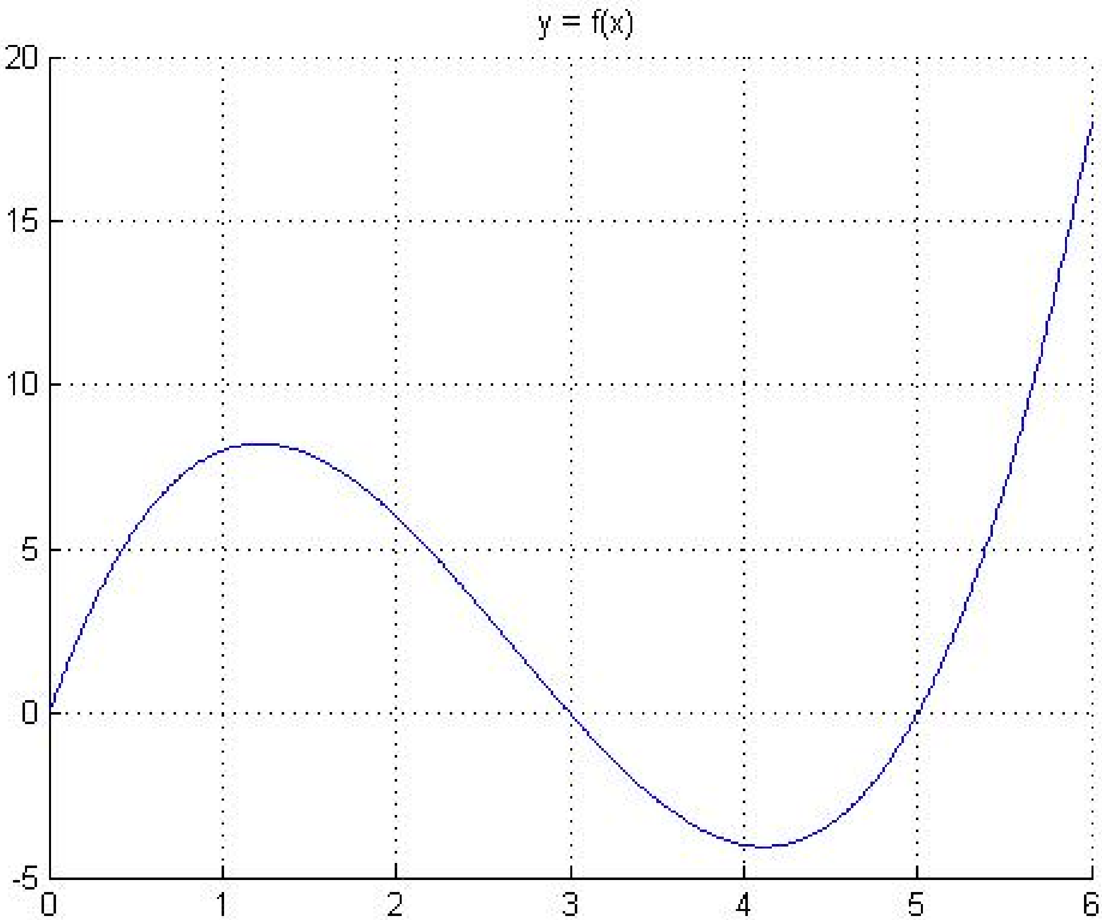
Previously we considered the derivative of a function, f, at a number a. But what if we let a vary ...

then,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now we have a new function, the derivative of $f(x)$

ex 35



try to sketch the graph of the derivative on the same graph thinking *in terms of slopes*

ex 36 Let $P(t)$ be the population of bacteria after t hours. Construct a table for $P'(t)$

t	$P(t)$
0	8,000
2	11,000
4	17,000
6	25,000

Let's approximate $P'(4)$ and the rest is similar...

$$P'(4) \;=\; \lim_{h\to 0} \frac{P(4+h)-P(4)}{h} \quad for\ small\ values\ of\ h$$

Well what values could we use for h? If we need to calculate $P(4+h)$ then what are our options? Looks like $h = 2$ or $h = -2$

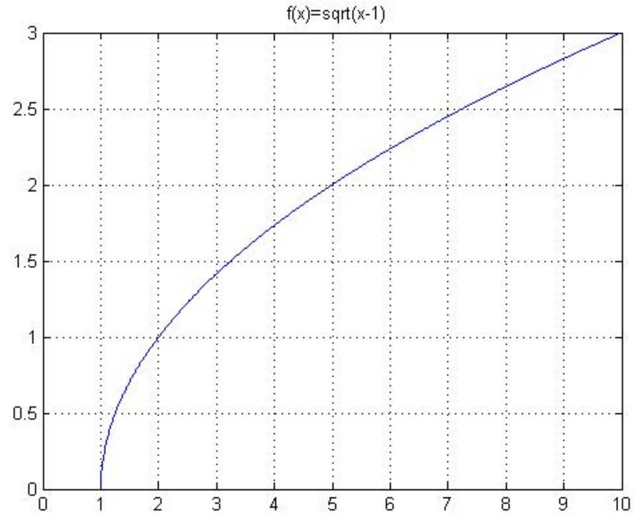
$$for\ \ h = 2 \quad P'(4) \;=\; \lim_{h\to 0} \frac{P(6)-P(4)}{2} \;=\; \frac{25000-17000}{2} \;=\; 4000$$

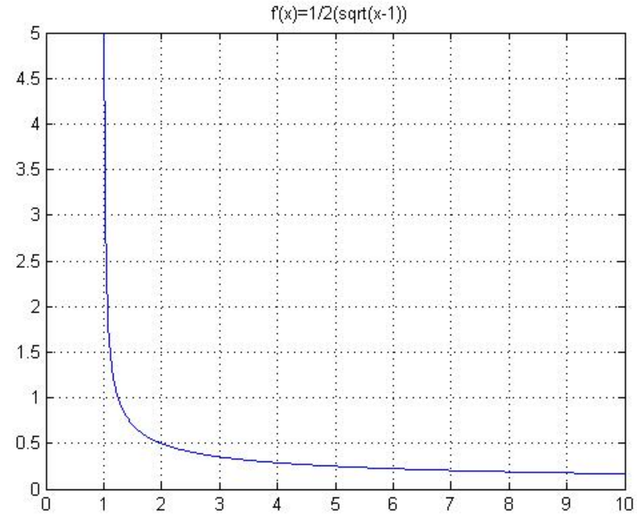
$$for\ \ h = -2 \quad P'(4) \;=\; \lim_{h\to 0} \frac{P(2)-P(4)}{-2} \;=\; \frac{11000-17000}{-2} \;=\; 3000$$

So our best guess would be what? $P'(4) \;\approx\; \frac{1}{2}(4000+3000) \;=\; 3500$

Continue in this fashion for the rest of the table

ex 37 If $f(x) = \sqrt{x-1}$, find $f'(x)$ and compare the graphs of both f and f'





Do these seem reasonable from a slope perspective??

Other Notations:

$$f'(x) \;=\; y' \;=\; \frac{dy}{dx} \;=\; Df(x) \;=\; D_x f(x)$$

Definition

A function f is *differentiable at* a if $f'(a)$ exists. It is differentiable on an open interval (a, b) if it is differentiable at every number in that interval.

ex 38 Where is $f(x) = \mid x \mid$ differentiable?

$$Note\ that \quad f'(x) \;=\; \lim_{h \to 0} \frac{\mid x+h \mid - \mid x \mid}{h}$$

$$If \;\; x > 0 \;\Longrightarrow\; \mid x+h \mid \;=\; x+h \;\; and \;\; \mid x \mid \;=\; x \;\Longrightarrow$$

$$f'(x) \;=\; \lim_{h \to 0} \frac{(x+h)-(x)}{h} \;=\; 1$$

So f is differentiable for $x > 0$

$$If \;\; x < 0 \;\; \Longrightarrow \;\; |\, x + h\,| \;\; = \;\; -\,(x+h) \;\; and \;\; |\, x\,| \;\; = \;\; -\,x \;\; \Longrightarrow$$

$$f'(x) \; = \; \lim_{h \to 0} \frac{-(x+h) + (x)}{h} \; = \; -\,1$$

So f is differentiable for $x < 0$

What about at $x = 0$?

$\lim_{h \to 0^+} \; f(x) \; = \; 1$
$\lim_{h \to 0^-} \; f(x) \; = \; -\,1 \qquad\qquad \Longrightarrow \quad f \; is \; NOT \; differentiable \; at \; 0$

Theorem

If f is differentiable at a then f is continuous at a

proof:

To show that f is continuous, we shall use the definition to show $\lim_{x \to a} \; f(x) = f(a)$

Observe

$$\begin{aligned} & \lim_{x \to a} \; f(x) \; = \; \lim_{x \to a} \; f(x) \\ \Longleftrightarrow \quad & \lim_{x \to a} \; f(x) \; = \; \lim_{x \to a} \; [f(x) + f(a) - f(a)] \\ \Longleftrightarrow \quad & \lim_{x \to a} \; f(x) \; = \; \lim_{x \to a} \; f(a) + \lim_{x \to a} \; (f(x) - f(a)) \end{aligned}$$

$$
\begin{aligned}
&\textit{Note however} \quad f(x)-f(a) = f(x)-f(a)\\
\iff\ & f(x)-f(a) = \frac{f(x)-f(a)}{x-a}(x-a)\\
\iff\ & \lim_{x\to a}\,(f(x)-f(a)) = \lim_{x\to a}\,\frac{f(x)-f(a)}{x-a}(x-a)\\
\iff\ & \lim_{x\to a}\,(f(x)-f(a)) = \lim_{x\to a}\,\left(\frac{f(x)-f(a)}{x-a}\right)\cdot\lim_{x\to a}\,(x-a)\\
&= f'(a)\cdot 0 \;=\; 0
\end{aligned}
$$

Thus

$$\lim_{x\to a}\,f(x) \;=\; \lim_{x\to a}\,f(a)+0 \;=\; f(a) \qquad \blacksquare$$

Is the converse of this theorem true? Does continuity $\Longrightarrow$ differentiability?

NO

Provide an example of a function that is continuous and not differentiable.

It turns out that there are three possible "issues" with a graph being differentiable.

1. sharp corners

2. discontinuities

3. vertical tangents

Worksheet for Section 7

1. Make a careful sketch of $f(x) = sin\ x$ and below it sketch $f'(x)$ thinking slopes. What is $f'(x)$?

2. Using the **DEFINITION** of the derivative, find $g'(x)$ if $g(x) = \frac{1}{x^2}$

Homework for Section 7

1. Make a careful sketch of the following functions and on the same set of axes sketch f'. Can you guess a formula for $f'(x)$ from the graphs?

   (a) $f(x) = sin\ x$

   (b) $f(x) = ln\ x$

   (c) $f(x) = e^x$

2. The following table gives displacements at various times t.

t	$s(t)$	t	$s(t)$
1	68	6	54
2	75	7	49
3	69	8	45
4	61	9	42
5	56	10	40

   (a) What is the meaning of $s'(t)$?

   (b) What are its units?

   (c) Construct a table of values for $s'(t)$.

3. SHOW where $f(x) =\mid x - 3 \mid$ is not differentiable.