MATH 151 Worksheet Keys Exam 3

Worksheet for Section 1

1. Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ has } a = \frac{1}{2} \text{ and } r = \frac{1}{2} \text{ so } \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 3\left(\sum\left(\frac{1}{i} - \frac{1}{i-1}\right)\right)$$

$$= \left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}-\tfrac{1}{3}\right) + \left(\tfrac{1}{3}-\tfrac{1}{4}\right) + \ldots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) = 1 - \tfrac{1}{n+1}$$

$$So \sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 3(1) + 1 = 4$$

2. Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^2+1} == \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 \neq 0$$

$$Thus \sum_{n=1}^{\infty} \frac{n^2}{n^2+1} \quad Diverges$$

Worksheet for Section 2

1. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

   $$= e \sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^{n-1} \quad \textit{which is a geometric series with } a = 1 \textit{ and } r = \frac{e}{3}$$

   $$\textit{Thus converges to} \quad \frac{e}{1-\frac{e}{3}} = \frac{3e}{3-e}$$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$

   $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = 1 \neq 0$$

   $$\textit{Thus} \quad \sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)} \quad \textit{Diverges}$$

2. Determine whether the series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

   $$\textit{Apply the Integral test} \quad \int_1^{\infty} \frac{x}{x^4+1}\, dx = \lim_{t\to\infty} \frac{1}{2} \int_1^{t} \frac{1}{u^2+1}\, du$$

   $$= \lim_{t\to\infty} \left[\frac{1}{2}\, tan^{-1} x^2 \Big|_1^t\right] = \lim_{t\to\infty} \frac{1}{2} \left[tan^{-1}(t^2) - tan^{-1}(1)\right] = \frac{\pi}{8}$$

   $$\textit{Thus} \quad \sum_{n=1}^{\infty} \frac{n}{n^4+1} \quad \textit{Converges}$$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$

   $$\textit{partial fractions yields} \quad \frac{3x+2}{x(x+1)} = \frac{2}{x} + \frac{1}{x+1} \quad \textit{then apply the Integral test}$$

   $$\int_1^{\infty} \left[\frac{2}{x} + \frac{1}{x+1}\right] dx = \lim_{t\to\infty} [2\, ln\, x + ln(x+1)] \Big|_1^t = \infty \ldots \sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)} \quad \textit{diverges}$$

Worksheet for Section 3

1. Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{ln\ n}{n^2}$

$$\frac{ln\ x}{x^2} \text{ is continuous, positive and decreasing so} \ldots$$

$$\int_2^{\infty} \frac{ln\ x}{x^2}\, dx = \ldots \text{ integration by parts } = \lim_{t\to\infty} \left(-\frac{ln\ x}{x} - \frac{1}{x} \right) \Bigg|_2^t = 1 \ by\ L'Hospitals\ \ldots\ converges$$

(b) $\displaystyle\sum_{n=1}^{\infty} ln\left(\frac{n}{2n+5}\right)$

$$\lim_{n\to\infty} ln\left(\frac{n}{2n+5}\right) = \lim_{n\to\infty} ln\left(\frac{1}{2+\frac{5}{n}}\right) = ln\frac{1}{2} \neq 0 \implies diverges$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{cos^2\ n}{n^2+1}$

$$\frac{cos^2\ n}{n^2+1} \leq \frac{1}{n^2+1} \leq \frac{1}{n^2} \quad and \quad \sum \frac{1}{n^2} \text{is a convergent p-series}$$

$$\implies \sum_{n=1}^{\infty} \frac{cos^2\ n}{n^2+1} \quad converges$$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

$$\text{do the limit comparison test with } \ b_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1 > 0$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}} \quad diverges$$

Worksheet for Section 4

1. Test the series for convergence or divergence.

   (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2n}{4n^2+1}$

   $$b_n=\frac{2n}{4n^2+1}>0 \ \textit{ and } b_n \textit{ is decreasing since } \ \frac{d}{dx}\left(\frac{2x}{4x^2+1}\right)<0 \ \textit{ for } x\geq 1$$

   $$\textit{so} \quad \lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{2n}{4n^2+1}=\lim_{n\to\infty}\frac{\frac{2}{n}}{4+\frac{1}{n^2}}=0$$

   $$\Longrightarrow \text{ convergence by the alternating series test}$$

   (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+\sqrt{n}}$

   $$b_n=\frac{\sqrt{n}}{1+\sqrt{n}}$$

   $$\textit{so} \quad \lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{\sqrt{n}}{1+\sqrt{n}}=\lim_{n\to\infty}\frac{1}{\frac{1}{\sqrt{n}}+1}=1\neq 0$$

   $$\Longrightarrow \textit{ diverges}$$

Worksheet for Section 5

1. Determine whether the series is absolutely convergent, conditionally convergent or divergent.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left| \left(\frac{(-10)^{n+1}}{(n+1)!}\right) \left(\frac{n!}{(-10)^n}\right) \right| = \lim_{n\to\infty} \left|\frac{-10}{n+1}\right| = 0 < 1$$

$$\implies \text{absolutely convergent}$$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+3)^n}{(3n+2)^n}$

$$\lim_{n\to\infty} \sqrt[n]{|\, a_n \,|} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n+3)^n}{(3n+2)^n}} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

$$\implies \text{absolutely convergent}$$

Worksheet for Section 6

1. Determine whether the series is absolutely convergent, conditionally convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

   $$\lim_{n\to\infty} a_n = \frac{1}{2} \implies \text{divergent}$$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

   $$\text{limit comparison test with} \frac{1}{n^{3/2}} \implies \text{convergent}$$

   (c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

   $$\text{integral or ratio test} \implies \text{convergent}$$

   (d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

   $$\text{alternating series test} \implies \text{convergent} \quad \text{CC}$$

   (e) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

   $$\text{ratio test} \implies \text{convergent} \quad \text{AC}$$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

   $$\text{limit comparison test with} \frac{1}{3^n} \implies \text{convergent}$$