MATH 151 Worksheets Exam 4

Worksheet for Section 1

1. Show that every member of the family of functions $y = Ce^{x^2/2}$ is a solution of the differential equation $y' = xy$.

$$y = C\ e^{x^2/2} \quad \Longrightarrow \quad y' = C\ e^{x^2/2}\left(\frac{2x}{2}\right) \ = \ xCe^{x^2/2} \ = \ xy$$

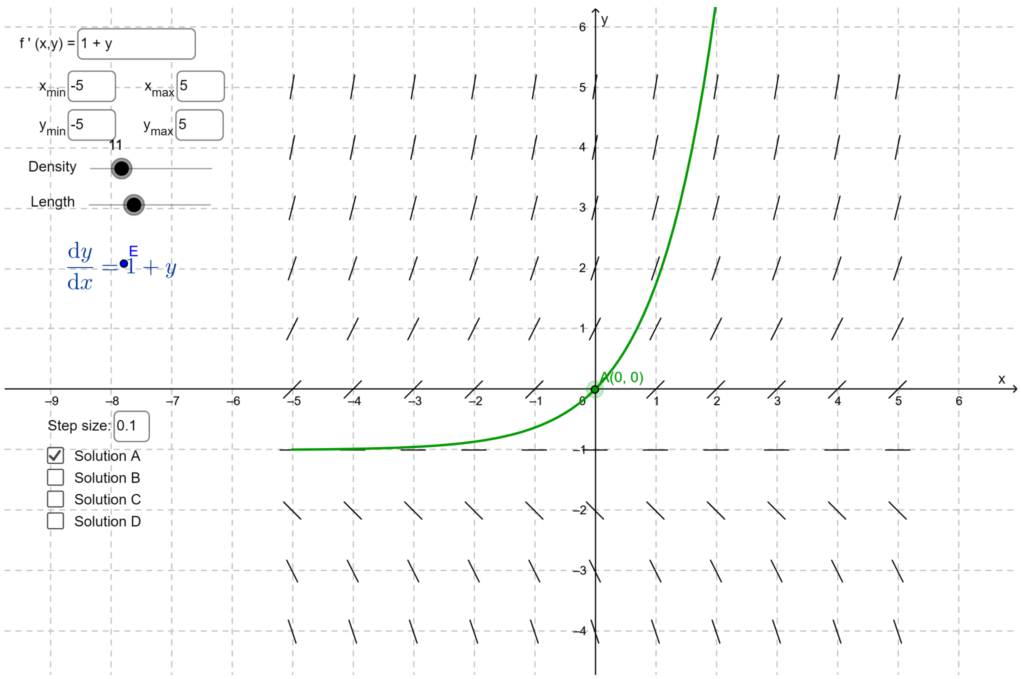
2. Find a solution of the differential equation $y' = xy$ that satisfies the initial condition $y(0) = 5$.

$$y(0) = 5 \quad \Longrightarrow \quad C\ e^0 = 5 \quad \Longrightarrow \quad C = 5$$

So the solution is $y = 5e^{x^2/2}$

Worksheet for Section 2

1. Sketch a direction field for the differential equation $y' = 1 + y$ and sketch a solution curve through the point $(0, 0)$.



2. Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial value problem $y' = 1 - xy$, $y(0) = 0$.

$$h = .2 \quad x_0 = 0 \quad y_0 = 0 \quad F(x, y) = 1 - xy$$

$$y_1 = y_0 + h\ F(x_0, y_0) = 0 + .2[1 - 0] = .2$$

$$y_2 = .2 + .2[1 - (.2)(.2)] = .392$$

$$y_3 = .56064$$

$$y_4 = .6933632$$

$$y_5 = .782425088 \qquad \Longrightarrow \qquad y(1) \approx .7824$$

Worksheet for Section 3

1. Solve the differential equation: $(x^2+1)y' = xy$

$$\frac{dy}{y} = \frac{x\ dx}{x^2+1} \implies \int \frac{1}{y}\ dy = \int \frac{x}{x^2+1}\ dx$$

$$\implies \quad ln|y| = \frac{1}{2} ln(x^2+1) + C \implies y = A\sqrt{x^2+1}$$

2. Solve the initial value problem: $x\ cos\ x = (2y + e^{3y})y'$, $y(0) = 0$

$$x\ cos\ x\ dx = (2y + e^{3y})dy \implies \int x\ cos\ x\ dx = \int (2y + e^{3y})\ dy$$

we need parts here $u = x$ $dv = cos\ x\ dx$ so we get $x\ sin\ x - \int sin\ x\ dx = x\ sin\ x + cos\ x + C$

$$\implies \quad y^2 + \frac{1}{3}e^{3y} = x\ sin\ x + cos\ x + C \text{ since } y(0) = 0 \implies C = -\frac{2}{3}$$

so the solution is $y^2 + \frac{1}{3}e^{3y} = x\ sin\ x + cos\ x - \frac{2}{3}$

Worksheet for Section 4

Show that if P satisfies the logistic equation, then

1. $\frac{d^2P}{dt^2} = k^2 P \left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{K}\right)$

$$\text{since we have that} \quad \frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

$$\frac{d^2P}{dt^2} = k\left[P\left(-\frac{1}{K}\frac{dP}{dt}\right) + \left(1 - \frac{P}{K}\right)\frac{dP}{dt}\right] = k\frac{dP}{dt}\left(-\frac{P}{K} + 1 - \frac{P}{K}\right)$$

$$= k\left[kP\left(1 - \frac{P}{K}\right)\right]\left(1 - \frac{2P}{K}\right) = k^2 P\left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{K}\right)$$

2. Show that a population grows fastest when it reaches half its carrying capacity.

the population grows the fastest when P' has a maximum, that is, when $P'' = 0$

$$P'' = 0 \quad \Longleftrightarrow \quad P = 0, K \text{ or } \frac{K}{2}$$

$$\text{since } 0 < P < K \quad \Longrightarrow \quad P'' = 0 \text{ when } P = \frac{K}{2}$$

Worksheet for Section 5

1. Solve the differential equation: $y' + 2y = 2e^x$

$$\rho(x) = e^{\int 2\ dx} = e^{2x}$$

$$\text{thus}\ \ e^{2x}y' + 2e^{2x}y = 2e^x e^{2x} \iff \left(e^{2x}y\right)' = 2e^{3x}$$

$$\text{so}\ \ e^{2x}y = \int 2e^{3x}\ dx \implies e^{2x}y = \frac{2}{3}e^{3x} + C \implies y = \frac{2}{3}e^x + Ce^{-2x}$$

2. Solve the initial value problem: $\frac{dv}{dt} - 2tv = 3t^2\ e^{t^2}$, $v(0) = 5$

$$\rho(t) = e^{\int -2t\ dt} = e^{-t^2}$$

$$\text{thus}\ \ e^{-t^2}\frac{dv}{dt} - 2te^{-t^2}v = 3t^2e^{t^2}e^{-t^2} \iff \left(e^{-t^2}v\right)' = 3t^2$$

$$\text{so}\ \ e^{t^2}v = \int 3t^2\ dt = t^3 + C \implies v = t^3e^{t^2} + Ce^{t^2}$$

$$\text{since}\ \ v(0) = 5 \implies C = 5\ \ \text{therefore}\ \ v = t^3e^{t^2} + 5e^{t^2}$$