

1 Integration by Parts

Recall that *integration* and *differentiation* are related. How?

Each differentiation rule has a corresponding integration rule

chain rule for differentiation \longleftrightarrow substitution for integration

product rule for differentiation \longleftrightarrow integration by parts

Recall the Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

if we let $u = f(x)$ and $v = g(x)$ then

$$\frac{d}{dx}[uv] = u dv + v du$$

integrating both sides gives us....

$$uv = \int u dv + \int v du$$

or

the formula for integration by parts:

$$\int u dv = uv - \int v du$$

or, using $f(x)$ and $g(x)$

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx$$

ex 1 find

$$\int x \sin x dx$$

Since

$$\int u dv = uv - \int v du$$

we need to choose **both** a u and a dv

the idea is to choose the u such that du gets "easier"

So let $u = x$, then $du = 1$, which makes $dv = \sin x$ and $v = -\cos x$

note that we can choose *any* antiderivative so just choose the simplest one, that is let $C = 0$

thus

$$\begin{aligned}\int x \sin x dx &= x(-\cos x) - \int (-\cos x) dx \\ &= -x \cos x + \int (\cos x) dx \\ &= -x \cos x + \sin x + C\end{aligned}$$

if we had let $u = \sin x$ and $dv = x dx$, we would have gotten

$$\int x \sin x dx = \frac{x^2}{2}(\sin x) - \frac{1}{2} \int x^2 \cos x dx$$

which clearly is NOT easier.

ex 2 Evaluate

$$\int \ln x \, dx$$

So let $u = \ln x$, then $dv = dx$, which makes $du = 1/x \, dx$ and $v = x$

this gives us

$$\dots \int \ln x \, dx = x \ln x - x + C$$

Sometimes you may need to do parts twice...

ex 3 Evaluate

$$\int x^2 \cos mx \, dx$$

So let $u = x^2$, then $dv = \cos mx \, dx$, which makes $du = 2x \, dx$ and $v = 1/m \sin mx$

thus

$$\int x^2 \cos mx \, dx = 1/m x^2 \sin mx - 2/m \int x \sin mx \, dx$$

Yes, this has gotten a little easier but we still need to evaluate $\int x \sin mx \, dx$

so use parts again with $U = x$, then $dV = \sin mx \, dx$, which makes $dU = dx$ and $V = -1/m \cos mx$

therefore

$$\begin{aligned} & \int x^2 \cos mx \, dx \\ &= 1/m x^2 \sin mx - 2/m[-1/m x \cos mx + 1/m^2 \sin mx + C] \\ &= 1/m x^2 \sin mx + 2/m^2 x \cos mx - 2/m^3 \sin mx + C \end{aligned}$$

This next example shows one final technique for integration by parts. You will notice that neither u or dv gets any easier but occasionally you can perform parts twice and get back the original function. When this happens you simply gather all the like terms, in this case the original integral, back over to the left hand side and simplify.

Observe:

ex 4 Evaluate

$$\int e^x \sin x \, dx$$

So let $u = e^x$, then $dv = \sin x \, dx$, which makes $du = e^x$ and $v = -\cos x$

so

$$(1) \quad \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

now, just looking at $\int e^x \cos x \, dx$

let $u = e^x$, then $dv = \cos x \, dx$, which makes $du = e^x$ and $v = \sin x$

so now we have

$$(2) \quad \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

and here you can see that we get back the original integral.

Combining (1) and (2) yields:

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

or

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x)$$

so it follows that

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

You can also use parts for Definite Integrals

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x) \, dx$$

Worksheet for Section 1

1. Evaluate using integration by parts:

(a) $\int \theta \sec^2 \theta d\theta$, let $u = \theta$, $dv = \sec^2 \theta d\theta$

(b) $\int x \cos 5x dx$

(c) $\int r e^{r/2} dr$

Homework for Section 1

1. Evaluate the following:

(a) $\int x^2 \ln x \, dx$ let $u = \ln x$ and $dv = x^2 \, dx$

(b) $\int x \cos 5x \, dx$

(c) $\int x e^{x/2} \, dx$

(d) $\int x^2 \sin \pi x \, dx$

(e) $\int \sin^{-1} x \, dx$

(f) $\int e^{2\theta} \sin 3\theta \, d\theta$

(g) $\int \cos x \ln(\sin x) \, dx$

2 Trigonometric Integrals

2.1 Integrals of the type

$$\int \sin^m x \cos^n x \, dx$$

ex 5 Evaluate

$$\int \sin^3 x \, dx$$

Clearly substitution will not work here

to integrate $\cos x$ you need a $\sin x$

to integrate $\sin x$ you need a $\cos x$

using $\cos^2 x + \sin^2 x = 1$ we have

$$\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx$$

then let $u = \cos x$ and $du = -\sin x$

so

$$\int \sin^3 x \, dx = \dots = -\cos x + \frac{1}{3}\cos^3 x + C$$

The idea is to write the integrand with either

1. One factor of sine and the rest cosine
or
2. One factor of cosine and the rest sine

ex 6 Evaluate

$$\int \sin^5 x \cos^2 x \, dx$$

rewriting...

$$\int \sin^5 x \cos^2 x \, dx = \int (\sin^2 x)^2 \cos^2 x \sin x \, dx$$

then substitution yields:

$$- \int (1 - u^2)^2 u^2 \, du = \dots = -\frac{1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x + C$$

As long as you have odd powers the preceding strategy is great but what if there are only even powers of sine and cosine?

Use the half-angle identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

or

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

ex 7 Find

$$\int \sin^4 x \, dx$$

rewriting...

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$$

and substituting the identities yields:

$$\begin{aligned}\int \left(\frac{1 - \cos 2x}{2} \right)^2 dx &= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) dx = \dots \\ &= \frac{1}{4} \left(\frac{3}{2} x - \sin 2x + \frac{1}{8} \sin 4x \right) + C\end{aligned}$$

So, the strategy for $\int \sin^m x \cos^n x dx$ is:

1. If the power of $\cos x$ is odd, save one $\cos x$ and make the rest $\sin x$ and let $u = \sin x$
2. If the power of $\sin x$ is odd, save one $\sin x$ and make the rest $\cos x$ and let $u = \cos x$
3. If both have even powers then use the half-angle identities

2.2 Integrals of the type

$$\int \tan^m x \sec^n x \, dx$$

Here the strategy is slightly different and you will be using the identity:

$$\sec^2 x = 1 + \tan^2 x$$

or

$$\tan^2 x = \sec^2 x - 1$$

1. If the power of $\sec x$ is even, save a $\sec^2 x$ and make the rest $\tan x$ and let $u = \tan x$
2. If the power of $\tan x$ is odd, save one $\sec x \tan x$ and make the rest $\sec x$ and let $u = \sec x$
3. Otherwise, be creative

ex 8 Find

$$\int_0^{\pi/4} \sec^4 \theta \tan^4 \theta \, d\theta$$

rewriting...

$$\int_0^{\pi/4} \sec^4 \theta \tan^4 \theta \, d\theta = \int_0^{\pi/4} (\tan^2 \theta + 1)(\tan^4 \theta) \sec^2 \theta \, d\theta$$

and using the substitution $u = \tan \theta$ and $du = \sec^2 \theta \, d\theta$ yields:

$$\int_0^{\pi/4} (u^2 + 1)(u^4) \, du = \dots = \frac{12}{35}$$

Worksheet for Section 2

1. Evaluate the integral:

(a) $\int \sin^6 x \cos^3 x \, dx$

(b) $\int_0^{\pi/2} \cos^2 \theta \, d\theta$

(c) $\int \tan^4 x \, dx$

Homework for Section 2

1. Evaluate the following:

$$(a) \int \sin^3 x \cos^2 x \, dx$$

$$(b) \int \sin^6 x \cos^3 x \, dx$$

$$(c) \int_0^{\pi/2} \cos^2 \theta \, d\theta$$

$$(d) \int (1 + \cos \theta)^2 \, d\theta$$

$$(e) \int \sec^2 x \tan x \, dx$$

$$(f) \int \tan^2 x \, dx$$

$$(g) \int \tan^3 x \sec x \, dx$$

3 Trigonometric Substitution

Now we will use a slightly different kind of substitution called *inverse substitution*

This method enables us to deal with integrals of the type

$$\int \sqrt{a^2 - x^2} \, dx$$

There are three kinds:

Expression	Substitution and Restriction	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta \quad -\pi/2 \leq \theta \leq \pi/2$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta \quad -\pi/2 < \theta < \pi/2$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta \quad 0 \leq \theta < \pi/2$	$\sec^2 \theta - 1 = \tan^2 \theta$

* Think of this as using trig and the triangle as a substitution *

ex 9 Evaluate

$$\int \frac{\sqrt{9 - x^2}}{x^2} \, dx$$

If we let $x = 3 \sin \theta$ then we have that $dx = 3 \cos \theta \, d\theta$ and

$\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2 \theta} = \sqrt{9 \cos^2 \theta} = 3 | \cos \theta | = 3 \cos \theta$ from the restriction.

Why can we substitute this way?

Think of a right triangle whose opposite side is x , adjacent is $\sqrt{9 - x^2}$

and whose hypotenuse is 3...

So, inverse substitution yields:

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2} dx &= \int \frac{3 \cos \theta}{9 \sin^2 \theta} 3 \cos \theta d\theta \\ &= \int \cot^2 \theta d\theta = \int (\csc^2 \theta - 1) d\theta \\ &= -\cot \theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1} \left(\frac{x}{3} \right) + C\end{aligned}$$

Note that we must return to the original x . We accomplish this by revisiting our triangle that was created for the substitution.

ex 10 Evaluate

$$\int_0^2 x^3 \sqrt{x^2+4} dx$$

If we let $x = 2 \tan \theta$ then we have that $dx = 2 \sec^2 \theta d\theta$

also note that we can change the bounds at this point. When $x = 0$, $\theta = 0$ and when $x = 2$, $\tan \theta = 1$ so $\theta = \pi/4$

Here we want to think of a right triangle whose opposite side is x , adjacent is 2 and whose hypotenuse is $\sqrt{x^2+4}$

So, inverse substitution yields:

$$\begin{aligned}\int_0^2 x^3 \sqrt{x^2+4} dx &= \int_0^{\pi/4} 2^3 \tan^3 \theta \sqrt{4 \tan^2 \theta + 4} 2 \sec^2 \theta d\theta \\ &= 2^5 \int_0^{\pi/4} \tan^3 \theta \sec \theta \sec^2 \theta d\theta = 2^5 \int_0^{\pi/4} (\sec^2 \theta - 1) \sec^2 \theta \sec \theta \tan \theta d\theta\end{aligned}$$

and if we let $u = \sec \theta$ and $du = \sec \theta \tan \theta d\theta$ then we have:

$$2^5 \int_0^{\pi/4} (u^4 - u^2) du = \dots = \frac{64}{15} (\sqrt{2} + 1)$$

Worksheet for Section 3

1. Evaluate the integral:

$$(a) \int \frac{1}{x^2 \sqrt{x^2 - 9}} dx \quad ; \quad x = 3 \sec \theta$$

$$(b) \int_{\sqrt{2}}^2 \frac{1}{t^3 \sqrt{t^2 - 1}} dt$$

Homework for Section 3

1. Evaluate the following:

$$(a) \int \frac{1}{x^2 \sqrt{x^2 - 9}} dx \quad x = 3 \sec \theta$$

$$(b) \int \frac{x^3}{\sqrt{x^2 + 9}} dx \quad x = 3 \tan \theta$$

$$(c) \int_{\sqrt{2}}^2 \frac{1}{x^3 \sqrt{x^2 - 1}} dx$$

$$(d) \int \frac{1}{x^2 \sqrt{25 - x^2}} dx$$

$$(e) \int \frac{dx}{\sqrt{x^2 + 16}}$$

$$(f) \int \sqrt{1 - 4x^2} dx$$

$$(g) \int \frac{x}{\sqrt{x^2 - 7}} dx$$

$$(h) \int \sqrt{5 + 4x - x^2} dx$$

4 Integration of Rational Functions using Partial Fractions

It should be routine to recall how to combine

$$\frac{2}{x-1} - \frac{1}{x+2} = \dots = \frac{x+5}{x^2+x-2}$$

Partial fraction decomposition reverses this process...

Why bother with this? Then it will be possible to integrate things like

$$\begin{aligned} & \int \frac{x+5}{x^2+x-2} dx \\ &= \dots = \int \frac{2}{x-1} dx - \int \frac{1}{x+2} dx \end{aligned}$$

In other words we can now integrate *Rational Functions*

Before we begin, the rational function in question must be *proper*, that is, the degree of the numerator must be less than the degree of the denominator.

If not, then you need to divide. Sometimes this is all that is necessary.

ex 11 Evaluate

$$\int \frac{x^3+x}{x-1} dx$$

This is an improper rational function so division gives us...

$$\frac{x^3 + x}{x - 1} = x^2 + x + 2 + \frac{2}{x - 1}$$

thus

$$\begin{aligned} \int \frac{x^3 + x}{x - 1} dx &= \int \left(x^2 + x + 2 + \frac{2}{x - 1} \right) dx \\ &= \dots = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln |x - 1| + C \end{aligned}$$

The focus will be on the denominator of the rational function as it **must** be factored as much as possible.

Fortunately there is a theorem that says ANY polynomial can be factored as a product of linear factors $(ax + b)$ and irreducible quadratic factors $(ax^2 + bx + c)$. As you may have guessed this will yield four cases...

4.1 CASE I: Distinct Linear Factors

So, by the previous theorem there must exist constants

$$A_1, A_2, \dots, A_k$$

such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_k}{a_kx + b_k}$$

ex 12 write the partial fraction decomposition of

$$\frac{1}{x^2 - 5x + 6}$$

so,

$$Q(x) = x^2 - 5x + 6 = (x - 3)(x - 2)$$

and we have two distinct linear factors

thus

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x - 3} + \frac{B}{x - 2}$$

multiply both sides of the equation by $Q(x)$ and get

$$1 = A(x - 2) + B(x - 3)$$

Now we have 2 options:

1. substitute values for x *or*

2. equate the coefficients

1. Since

$$1 = A(x - 2) + B(x - 3)$$

$$x = 3 \implies A = 1$$

$$x = 2 \implies B = -1$$

or

2. Since

$$1 = A(x - 2) + B(x - 3)$$

$$1 = (A + B)x - 2A - 3B$$

$$A + B = 0 \implies A = -B$$

$$\text{also } -2A - 3B = 1 \implies A = 1 \text{ and } B = -1$$

It depends on the problem as to which method is easier. The first method usually works better on linear factors and the second usually works better with quadratic factors.

ex 13 Evaluate

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$$

so

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

since we have 3 distinct linear factors. Expanding gives us

$$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

thus

$$\begin{aligned} 2A + B + 2C = 1 & \implies A = \frac{1}{2} \\ 3A + 2B - C = 2 & \implies B = \frac{1}{5} \\ -2A = -1 & \implies C = -\frac{1}{10} \end{aligned}$$

therefore

$$\begin{aligned} \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx &= \int \left[\frac{1}{2} \frac{1}{x} + \frac{1}{5} \frac{1}{2x - 1} - \frac{1}{10} \frac{1}{x + 2} \right] dx \\ &= \dots = \frac{1}{2} \ln |x| + \frac{1}{10} \ln |2x - 1| - \frac{1}{10} \ln |x + 2| + C \end{aligned}$$

4.2 CASE II: Repeated Linear Factors

Suppose that a linear factor, $(a_1x + b_1)$ is repeated r times, that is, $(a_1x + b_1)^r$ is in the factorization of $Q(x)$.

Then instead of

$$\frac{A_1}{a_1x + b_1}$$

you will need

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

ex 14 Evaluate

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} dx$$

so

$$x^3 + 2x^2 + x = x(x + 1)^2$$

thus

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$$

and

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx$$

$$\text{If } x = 0 \quad \implies \quad A = 6$$

$$\text{If } x = -1 \quad \implies \quad C = 9$$

to find B use any other value of x along with $A = 6$ and $C = 9$

$$\text{using } x = 1 \quad \Longrightarrow \quad 31 = 4A + 2B + C \quad \Longrightarrow \quad B = -1$$

therefore

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} dx = \int \left[\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2} \right] dx$$

$$= \dots = \ln \left| \frac{x^6}{x+1} \right| - \frac{9}{x+1} + C$$

NOTE:

you must make as many substitutions as there are unknowns with this method. We used $x = 0, 1$ and 2 .

4.3 CASE III: Distinct Linear and Irreducible Quadratic Factors

Now you will have a term of the type:

$$\frac{Ax + B}{ax^2 + bx + c}$$

Also, you will need the following formula:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

ex 15 Evaluate

$$\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx$$

so

$$(x^2 - x)(x^2 + 4) = x(x - 1)(x^2 + 4)$$

thus

$$\frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 4}$$

and

$$2x^3 - 4x - 8 = A(x - 1)(x^2 + 4) + Bx(x^2 + 4) + (Cx + D)x(x - 1)$$

$$\text{If } x = 0 \quad \implies \quad A = 2$$

$$\text{If } x = 1 \quad \implies \quad B = -2$$

$$\text{If } x = -1, A = 2, B = -2 \quad \implies \quad C = 2$$

$$\text{If } x = 2, A = 2, B = -2 \quad \implies \quad D = 4$$

therefore

$$\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx = \int \left[\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2+4} + \frac{4}{x^2+4} \right] dx$$
$$= \dots = 2 \ln |x| - 2 \ln |x-1| + \ln(x^2+4) + 2 \tan^{-1} \left(\frac{x}{2} \right) + C$$

4.4 CASE IV: Repeated Linear and Irreducible Quadratic Factors

If $Q(x)$ has a factor $(ax^2 + bx + c)^r$ then the partial fraction decomposition must have:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Case IV will require you to equate coefficients to solve for the constants

ex 16 Evaluate

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx$$

so

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}$$

and

$$\begin{aligned} 8x^3 + 13x &= (Ax + B)(x^2 + 2) + Cx + D \\ \implies 8x^3 + 13x &= Ax^3 + Bx^2 + (2A + C)x + (2B + D) \end{aligned}$$

$$\implies A = 8$$

$$\implies B = 0$$

$$2A + C = 13 \implies C = -3$$

$$2B + D = 0 \implies D = 0$$

therefore

$$\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx = \int \left[\frac{8x}{x^2 + 2} - \frac{3x}{(x^2 + 2)^2} \right] dx$$
$$= \dots = 4 \ln(x^2 + 2) + \frac{3}{2(x^2 + 2)} + C$$

There are times when some non-rational functions can be changed into rational functions by a clever substitution.

ex 17 Evaluate

$$\int \frac{\sqrt{x+4}}{x} dx$$

If we let $u = \sqrt{x+4}$, then $u^2 = x+4$, $x = u^2 - 4$ and $dx = 2u du$
thus

$$\int \frac{\sqrt{x+4}}{x} dx = \int \frac{u}{u^2 - 4} 2u du =$$

after dividing the rational function

$$= 2 \int \left[1 + \frac{4}{u^2 - 4} \right] du$$

and then doing partial fractions on $u^2 - 4 = (u+2)(u-2)$ we get...

$$\dots = 2u + 2(\ln |u-2| - \ln |u+2|) + C$$
$$= 2\sqrt{x+4} + 2(\ln |\sqrt{x+4}-2| - \ln |\sqrt{x+4}+2|) + C$$

it is not always necessary to use partial fractions on all rational functions

ex 18 Evaluate

$$\begin{aligned} & \int \frac{x^2 + 1}{x^3 + 3x - 4} dx \\ &= \frac{1}{3} \int \frac{3x^2 + 3}{x^3 + 3x - 4} dx = \int \frac{1}{u} du \\ &= \frac{1}{3} \ln |x^3 + 3x - 4| + C \end{aligned}$$

ex 19 Find

$$\begin{aligned} & \int \frac{x^2 - x - 2}{x^3 - 2x - 4} dx \\ &= \int \frac{(x + 1)(x - 2)}{(x - 2)(x^2 + 2x + 2)} dx = \int \frac{x + 1}{x^2 + 2x + 2} dx \\ &= \frac{1}{2} \ln |x^2 + 2x + 2| + C \end{aligned}$$

Worksheet for Section 4

Section 4 Worksheet Case I/II

1. Evaluate the integral:

$$(a) \int \frac{x - 9}{(x + 5)(x - 2)} dx$$

$$(b) \int \frac{1}{(x + 5)^2(x - 1)} dx$$

Section 4 Worksheet Case III/IV

1. Evaluate the integral:

$$(a) \int \frac{2x^2 - x + 4}{x^3 + 4x} dx$$

$$(b) \int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} dx$$

Homework for Section 4

1. Evaluate the following:

$$(a) \int \frac{x}{x-6} dx$$

$$(b) \int \frac{x-9}{(x+5)(x-2)} dx$$

$$(c) \int_2^3 \frac{1}{x^2-1} dx$$

$$(d) \int \frac{1}{(x+5)^2(x-1)} dx$$

$$(e) \int \frac{5x^2+3x-2}{x^3+2x^2} dx$$

2. Make a substitution to obtain a rational function and then evaluate the following:

$$(a) \int \frac{1}{x\sqrt{x+1}} dx$$

$$(b) \int \frac{x^3}{\sqrt[3]{x^2+1}} dx$$

5 Integration Strategies

If you do not immediately see how to integrate, try this 4 step process:

1. Simplify the integrand

$$\int (\sin x + \cos x)^2 dx = \dots = \int (1 + 2 \sin x \cos x) dx = \dots$$

2. Look for the substitution

$$\int \frac{x}{x^2 - 1} dx$$

Here partial fractions is **NOT** necessary

3. Try to classify the integrand:

- (a) trig functions \longrightarrow use trig substitution
- (b) rational functions \longrightarrow use partial fractions
- (c) (polynomial)(transcendental) \longrightarrow use integration by parts
- (d) radicals in the integrand \longrightarrow use trig substitution or clever substitution ($u = \sqrt{\text{stuff}}$)

4. If all else fails just realize there are really only two methods: substitution and parts

- (a) try a clever substitution or
- (b) try integration by parts or
- (c) try manipulation using identities, rationalization, etc...

ex 20 Find

$$\int \frac{dx}{1 - \cos x}$$

$$\begin{aligned} \int \frac{dx}{1 - \cos x} &= \int \left(\frac{1}{1 - \cos x} \right) \left(\frac{1 + \cos x}{1 + \cos x} \right) dx \\ &= \int \frac{1 + \cos x}{\sin^2 x} dx \\ &= \int \left(\csc^2 x + \frac{\cos x}{\sin^2 x} \right) dx \\ &= \dots \end{aligned}$$

Worksheet for Section 5

1. Evaluate the integral:

$$(a) \int_0^2 \frac{2t}{(t-3)^2} dt$$

$$(b) \int_{-1}^1 \frac{e^{\arctan y}}{1+y^2} dy$$

$$(c) \int_1^3 r^4 \ln r dr$$

$$(d) \int \sin^3 \theta \cos^5 \theta d\theta$$

Homework for Section 5

1. Evaluate the following:

$$(a) \int \frac{x}{\sqrt{3-x^4}} dx$$

$$(b) \int \frac{dx}{(1-x^2)^{3/2}}$$

$$(c) \int e^{x+e^x} dx$$

$$(d) \int_0^1 (1+\sqrt{x})^8 dx$$

$$(e) \int \frac{3x^2-2}{x^2-2x-8} dx$$

$$(f) \int \sqrt{\frac{1+x}{1-x}} dx$$

$$(g) \int \sqrt{3-2x-x^2} dx$$

$$(h) \int_0^{\pi/4} \cos^2 \theta \tan^2 \theta d\theta$$

$$(i) \int e^x \sqrt{1+e^x} dx$$