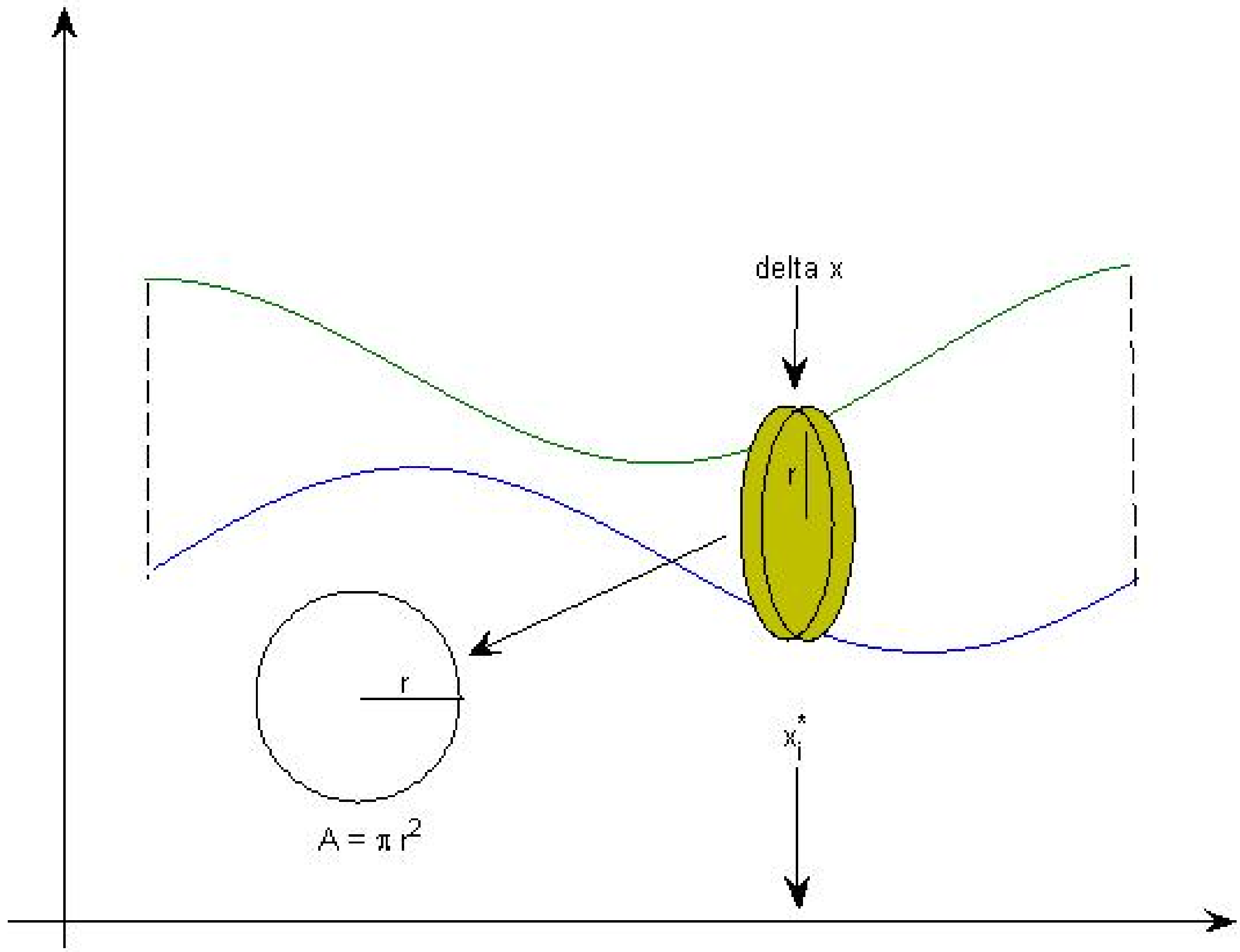
1 Volumes



The key to calculating the volume of an object is to first find the area of a *cross-section*

Recall that to estimate the area under the curve we approximated with rectangles then took a limit

A similar concept is at work here. Now we will estimate the volume with disks and take a limit as the number of disks approaches infinity

That is, the volume of the disk above is $A(x_i^*)\Delta x$

Definition

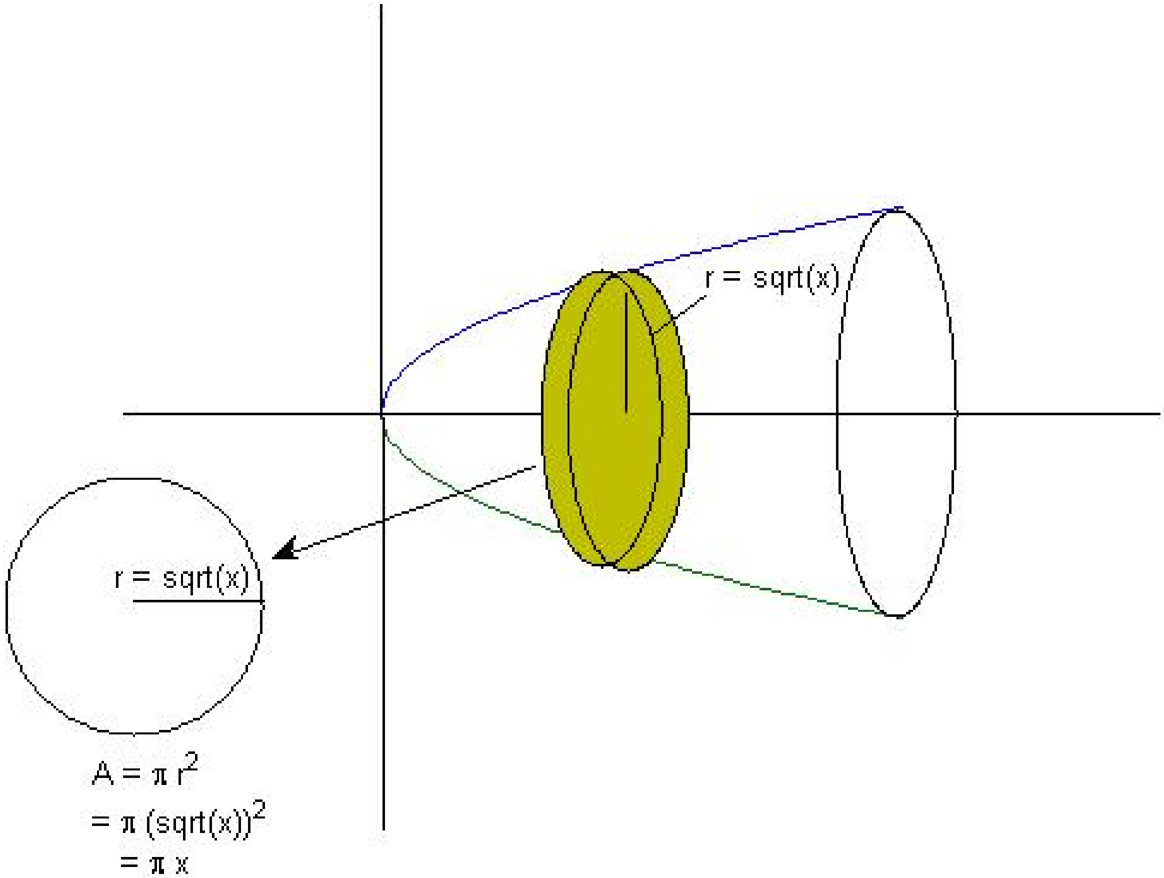
Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area through x, perpendicular to the x-axis is $A(x)$, where A is a continuous function, then

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx$$

ex 1 Find the volume of the solid obtained by rotating $y = \sqrt{x}$ about the x-axis from 0 to 1.

I highly recommend you do the following for volume questions:

1. *sketch a picture*
2. *sketch a sample disk next to the picture*
3. *find the area of your sample disk*
4. *set up the integral and solve*

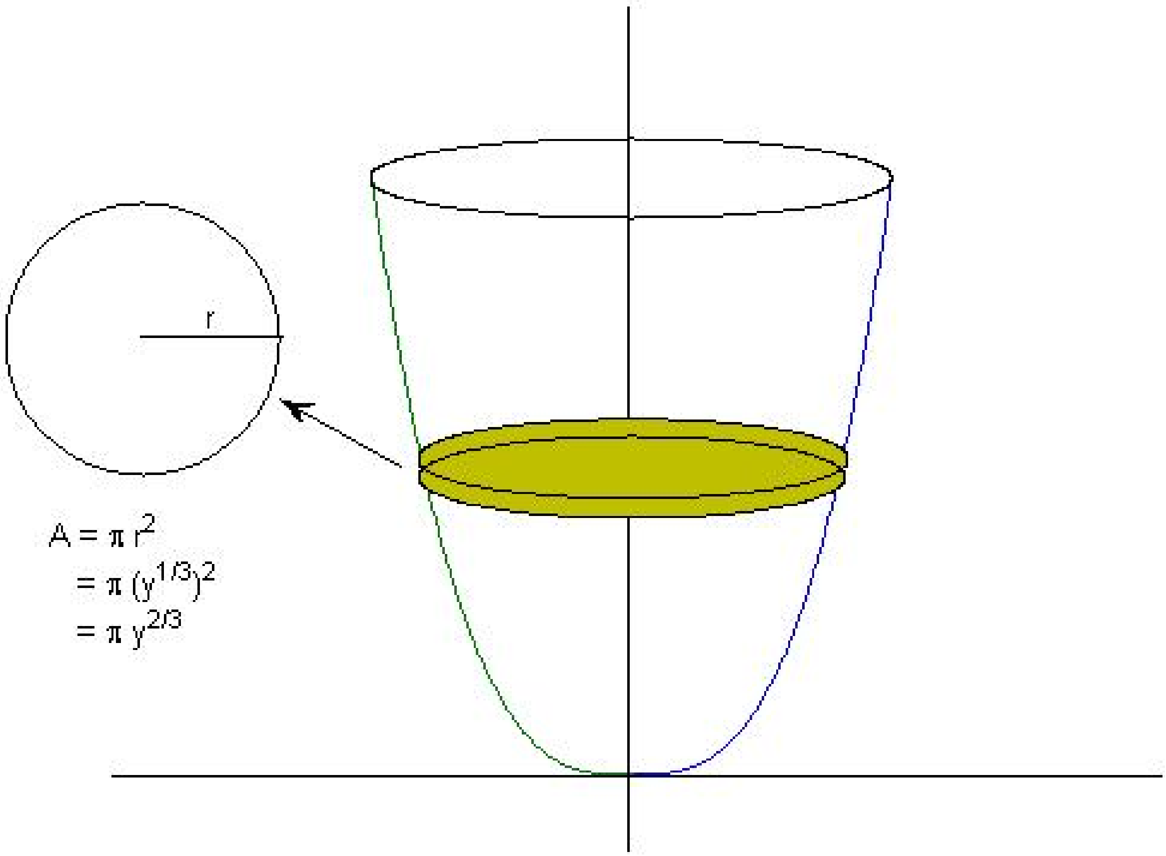


So

$$\text{Since the area is given above we have } V = \int_a^b A(x)\,dx$$

$$= \int_0^1 \pi(x)\,dx = \ldots = \frac{\pi}{2}$$

ex 2 Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.

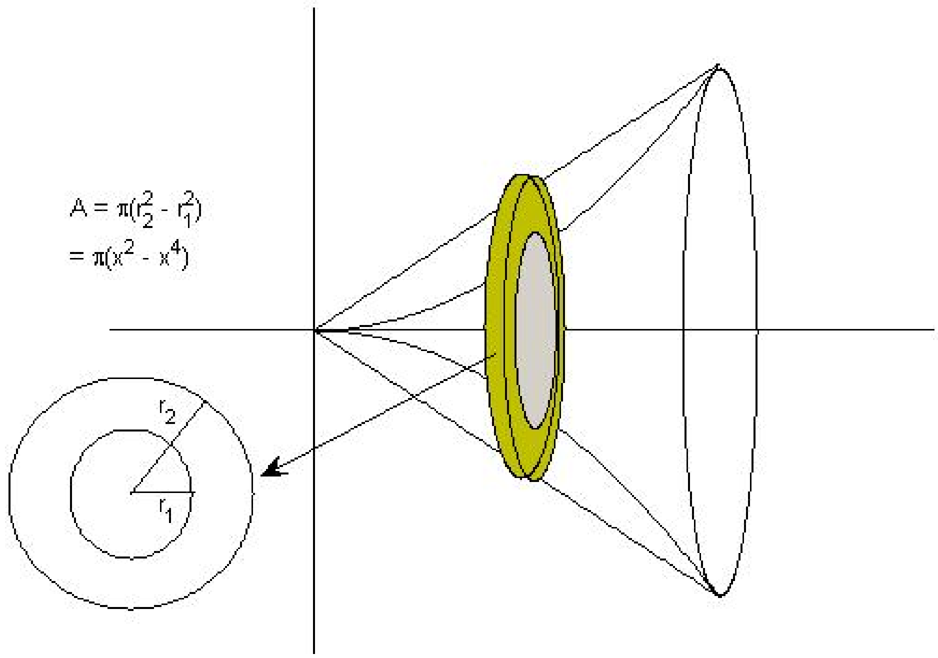


Since this region is rotated about the y-axis, we will slice our disk perpendicular to the y-axis and integrate with respect to y

Since the area is given above we have $V \; = \; \int_a^b A(y)\, dy$

$$= \int_0^8 \pi(y)^{2/3}\, dy \; = \; \pi\left({}^3\!/_5\right) y^{5/3}\,\Bigg|_0^8 \; = \; \frac{96\pi}{5}$$

ex 3 A region R in enclosed by $y = x$ and $y = x^2$ and rotated about the x-axis. Find the volume of the resulting solid.



Since this region has some space between the curve and the x-axis, when we rotate it we will not get disks. We will obtain washers.

Since the area is given above we have $V = \int_a^b A(x)\,dx$

$$= \int_0^1 \pi(x^2 - x^4)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]\Bigg|_0^1 = \frac{2\pi}{15}$$

Worksheet for Section 1

1. Find the volume of the solid obtained by rotating the region bounded by $y = e^x,\ y = 0,\ x = 0,\ x = 1$ about the x-axis. Sketch the region and a typical disk or washer.

2. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x-1},\ x = 2,\ x = 5,\ y = 0$ about the x-axis. Sketch the region and a typical disk or washer.

3. Find the volume of the solid obtained by rotating the region bounded by $x = y - y^2,\ x = 0$ about the y-axis. Sketch the region and a typical disk or washer.

Homework for Section 1

1. Find the volume by rotating the following regions bounded by the curves, about the specified lines. Sketch the region, the solid and a typical disk or washer.

   (a) $y = 1/x\ ,\quad x = 1\ ,\quad x = 2\ ,\quad y = 0\quad$ about the $x - axis$

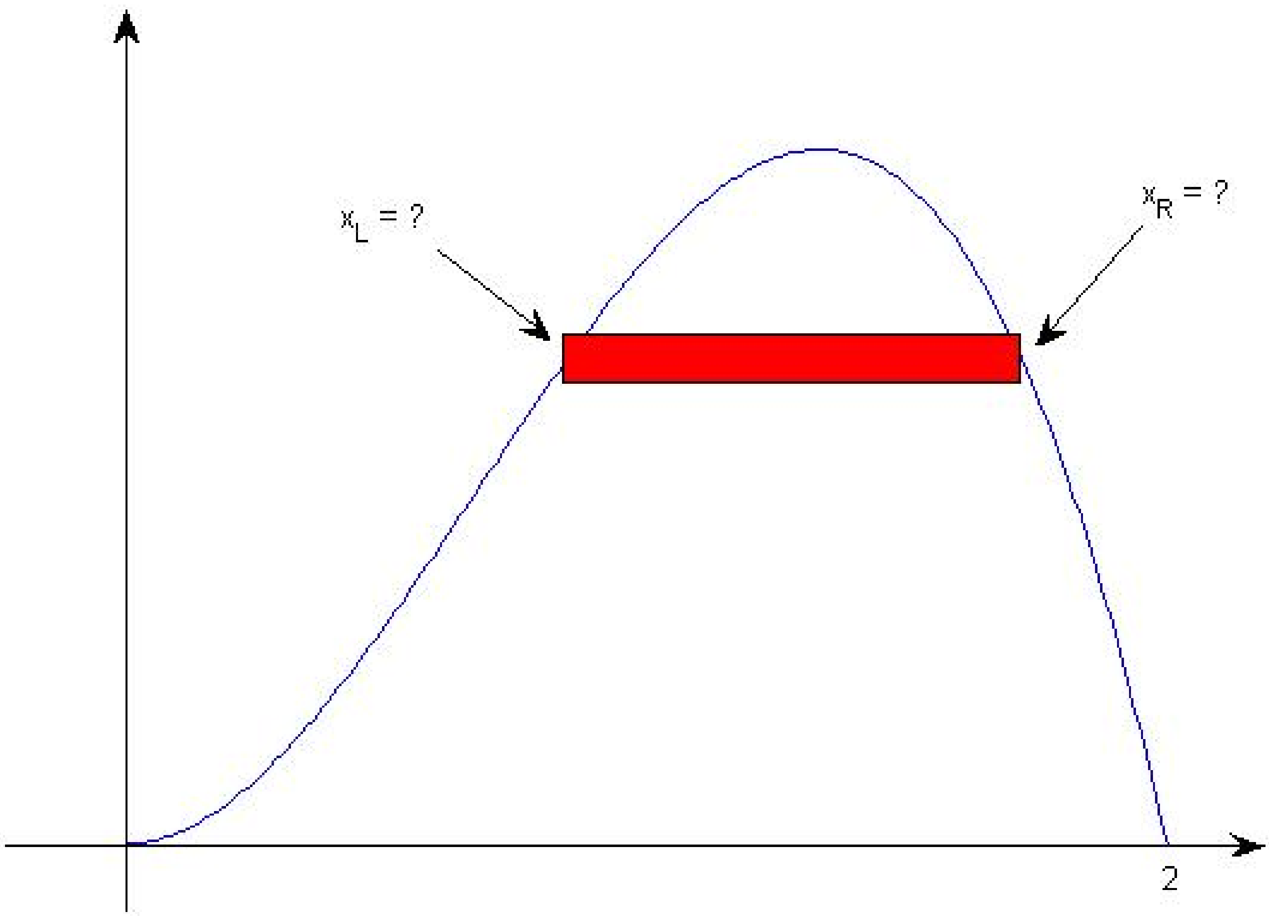
   (b) $x = 2\sqrt{y}\ ,\quad x = 0\ ,\quad y = 9\quad$ about the $y - axis$

   (c) $y = x^3\ ,\quad y = x\ ,\quad x \geq 0\quad$ about the $x - axis$

   (d) $y^2 = x\ ,\quad x = 2y\quad$ about the $y - axis$

2 Volume by Cylindrical Shells

ex 4 Find the volume by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis



First, we would need to solve $y = 2x^2 - x^3$ for x. Then we would need a local max.

There is a MUCH easier way ...

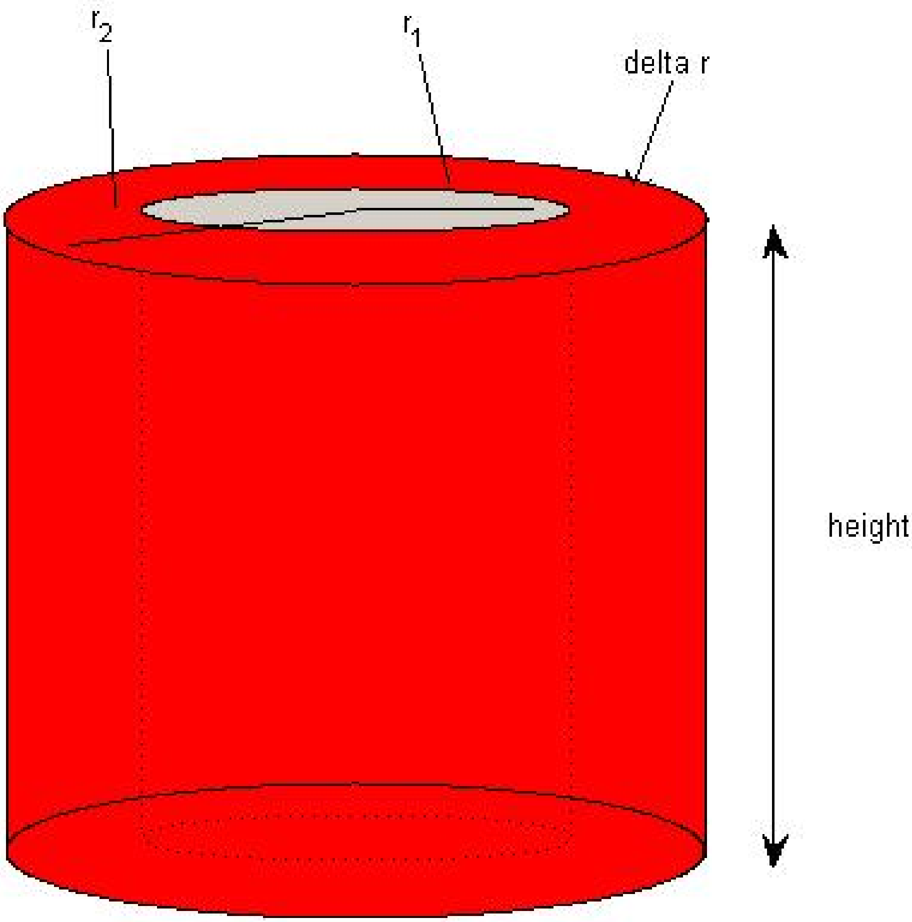
What if we used shells to approximate the volume instead of disks or washers?

What is a shell?

A shell is simply a hollow cylinder

We will need the volume of this shell.

Observe



$$
\begin{aligned}
V &= V_2 - V_1 \\
&= \pi(r_2)^2 h - \pi(r_1)^2 h \\
&\vdots
\end{aligned}
$$

$$\begin{aligned} V &= 2\pi r h \Delta r \\ &= (\text{circumference})(\text{height})(\text{thickness}) \end{aligned}$$

Definition

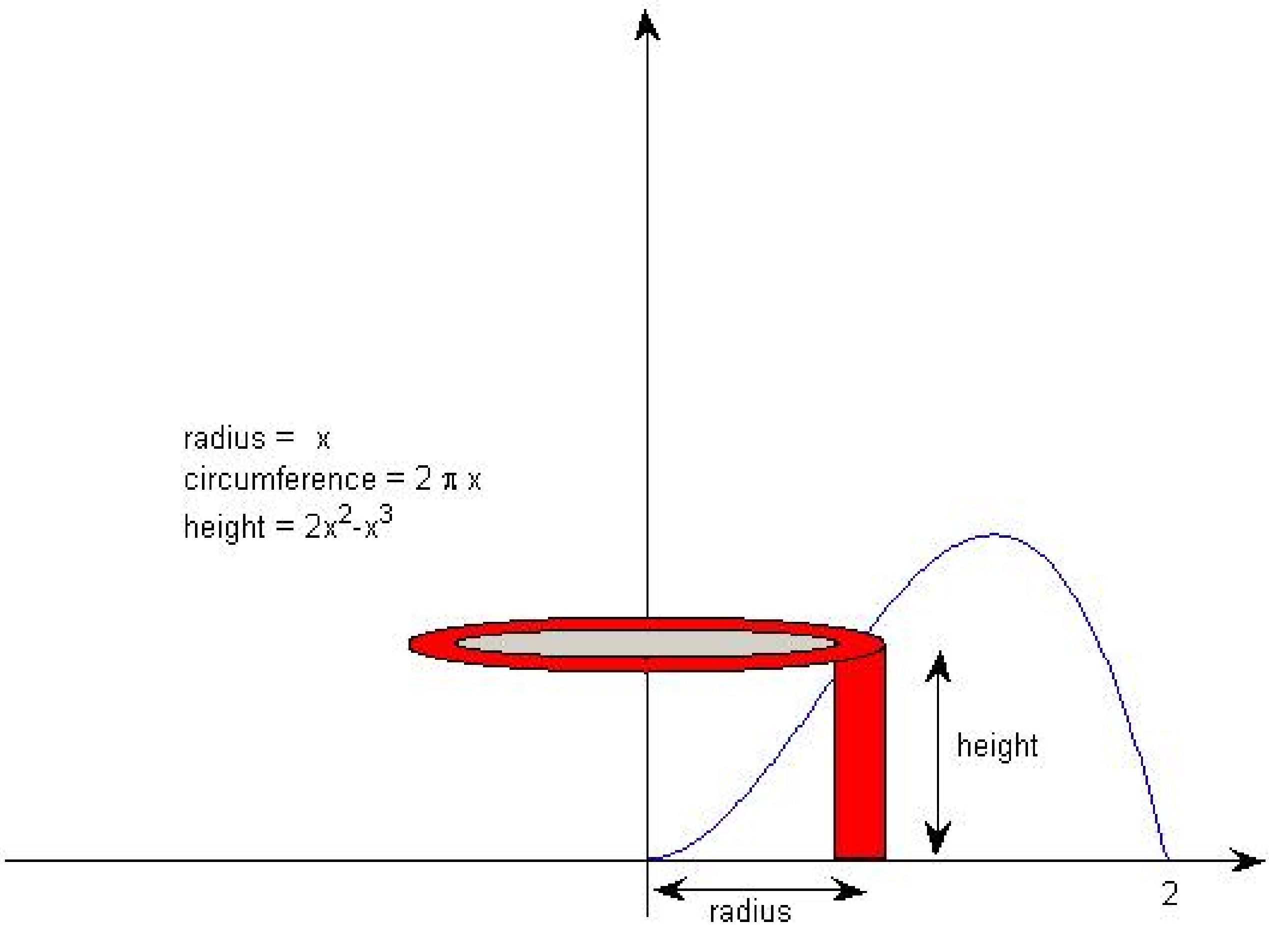
The volume of a solid obtained by rotating the region under the curve $y = f(x)$ from $x = a$ to $x = b$ about the y-axis is:

$$\int_a^b 2\pi x f(x)\ dx$$

Where

$$2\pi x \;=\; \text{circumference} \qquad \text{and} \qquad f(x) \;=\; \text{height}$$

So, back to the original example ...

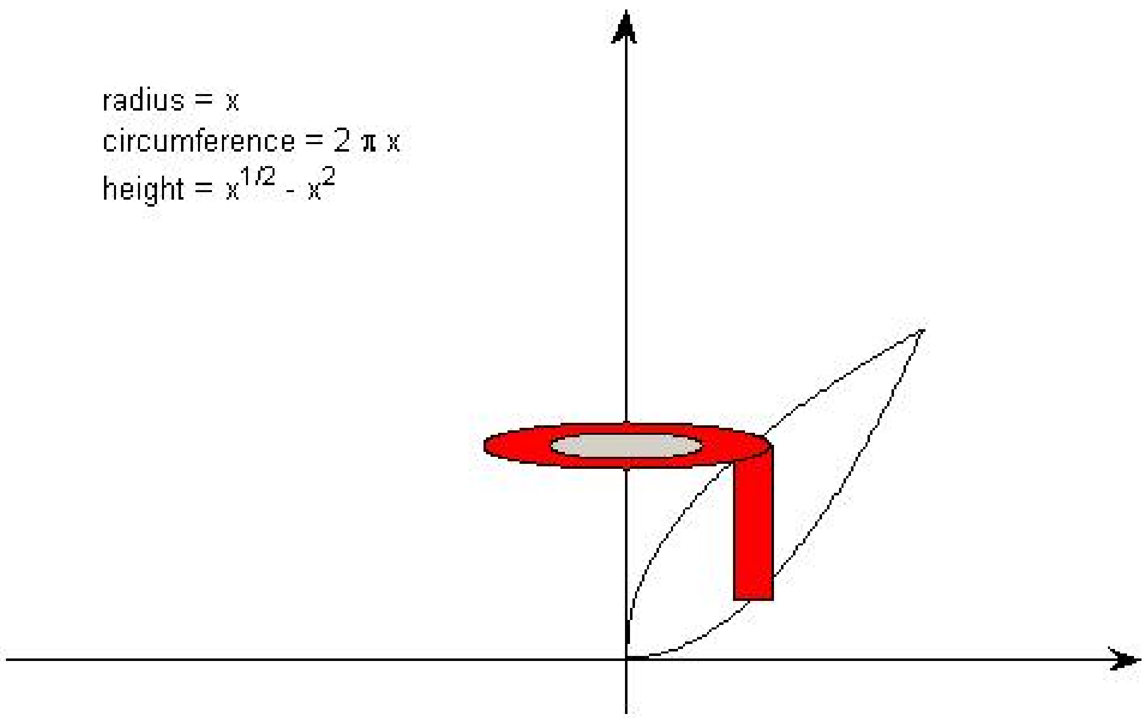


$$\Longrightarrow \quad V \; = \; \int_0^2 2\pi x(2x^2 - x^3)\; dx \; =$$

$$2\pi \int_0^2 (2x^3 - x^4)\; dx$$

$$= \; 2\pi \left(\frac{x^4}{2} - \frac{x^5}{5} \right) \Bigg|_0^2 \; = \; \ldots \; = \; \frac{16\pi}{5}$$

ex 5 Find the volume by rotating the region bounded by $y = \sqrt{x}$ and $y = x^2$ about the y-axis. Use shells.

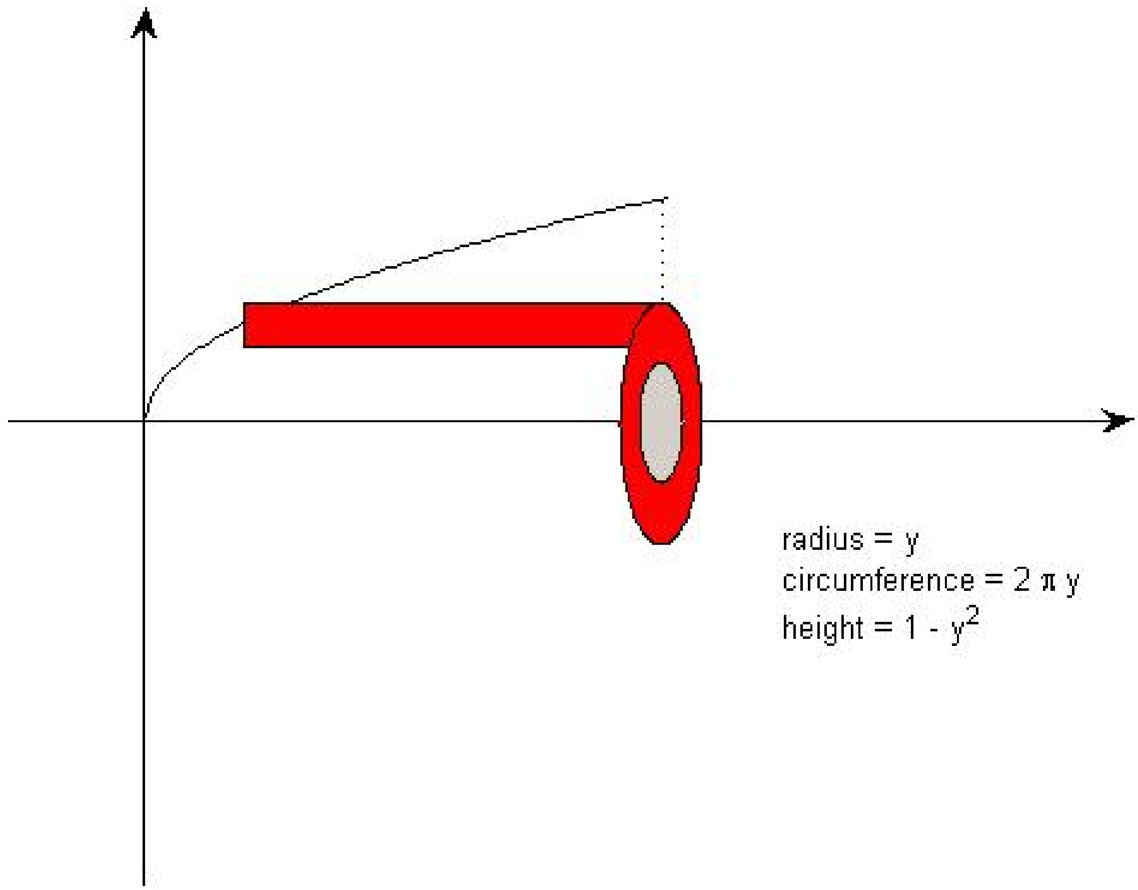


$$\Longrightarrow \quad V \;=\; \int_0^1 2\pi x(\sqrt{x} - x^2)\; dx \;=$$

$$2\pi \int_0^1 (x^{3/2} - x^3)\; dx$$

$$=\; 2\pi\left(\frac{2x^{5/2}}{5} - \frac{x^4}{4}\right)\Bigg|_0^1 \;=\; \ldots \;=\; \frac{3\pi}{10}$$

ex 6 Find the volume by rotating the region bounded by $y = \sqrt{x}$ about the x-axis from 0 to 1.



$$\Longrightarrow \quad V \;=\; \int_0^1 2\pi y(1-y^2)\;dy \;=$$

$$2\pi \int_0^1 (y-y^3)\;dy$$

$$=\; 2\pi\left(\frac{y^2}{2}-\frac{y^4}{4}\right)\Bigg|_0^1 \;=\; \ldots \;=\; \frac{\pi}{2}$$

What about using disks?

$$A(x) \;=\; \pi(\sqrt{x})^2 \;=\; \pi x \quad \Longrightarrow \quad V \;=\; \int_0^1 A(x)\;dx \;=$$

$$\int_0^1 \pi x \ dx$$

$$= \pi \left(\frac{x^2}{2} \right) \Big|_0^1 = \frac{\pi}{2}$$

Which do you think is easier?

Worksheet for Section 2

1. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = x(x-1)^2$ and the x-axis about the y-axis. Sketch the region and a typical shell.

2. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $x = 4y^2 - y^3$ and the line $x = 0$ about the x-axis. Sketch the region and a typical shell.

Homework for Section 2

1. Find the volume by rotating the following regions bounded by the curves, about the specified lines. Sketch the region, the solid and a typical shell.

   (a) $y = 1/x$, $x = 1$, $x = 2$, $y = 0$ about the $y - axis$

   (b) $y = e^{-x^2}$, $x = 0$, $x = 1$, $y = 0$ about the $y - axis$

   (c) $1 + y^2 = x$, $x = 0$, $y = 1$, $y = 2$ about the $x - axis$

   (d) $y = x^3$, $x = 0$, $y = 8$ about the $x - axis$

   (e) $y = 4x - x^2$, $y = 3$ about the line $x = 1$

2. Set up but DO NOT EVALUATE the solid obtained by rotating the region bounded by $y = ln\ x$, $y = 0$ and $x = 2$ about the y-axis.

3 Average Value of a Function

If you have n numbers and you would like their average, then ...

$$x_{avg} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

How about the average temperature during a day? Is there a finite number of time intervals?

This is the equivalent of finding the average value of a function since the function takes on how many values?

Definition

The average value of a function f on an interval $[a, b]$ is:

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

ex 7 Find the average value of $y = 1 + x^2$ on $[-1, 2]$.

$$f_{avg} = \frac{1}{2-(-1)} \int_{-1}^{2} (1 + x^2)\, dx = \ldots = 2$$

Mean Value Theorem for Integrals

If f is continuous on $[a, b]$ then there exists a number c in $[a, b]$ such that

$$f(c) \ = \ f_{avg} \ = \ \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is

$$\int_a^b f(x)\,dx \ = \ f(c)(b-a)$$

Worksheet for Section 3

1. Find the average value of $h(r) = \frac{3}{(1+r)^2}$ on the interval $[1, 6]$.

Homework for Section 3

1. Find the average value of the function on the given interval.

   (a) $f(x) = \sqrt[3]{x}\ , \quad [1, 8]$

   (b) $f(x) = xe^{-x^2}\ , \quad [0, 5]$

   (c) $f(x) = cos^4\ x\ sin\ x\ , \quad [0, \pi]$

2. For the function $f(x) = (x - 3)^2$ on $[2, 5]$:

   (a) find the average value

   (b) find c such that $f_{avg} = c$

   (c) sketch f and a rectangle whose area is the same as the area under f

4 Arc Length

How would you calculate the length of a curve?

As we have done many times before, use something simple to approximate something complex.

Divide the curve into a finite number of straight line segments, which are easier to calculate.

As the number of segments increases, your approximation gets better and better.

If we let $\mid P_{i-1}\ P_i \mid$ represent the length of each line segment then

the length of the curve is equal to the limit of the lengths of these segments. That is,

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \left| P_{i-1}\ P_i \right|$$

This definition does not really help us with the computation though.

However, we can derive an integral formula in the case that f has a continuous derivative, that is, the case when f is smooth.

$$\mid P_{i-1}\ P_i \mid \ = \ \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \ = \ \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

Apply the MVT to f on $[x_{i-1}, x_i]$, thus there exists an x_i^* in $[x_{i-1}, x_i]$ such that $f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$

or

$\Delta y_i = f'(x_i^*)\Delta x$

Thus

$$| \; P_{i-1} \, P_i \; | \;\; = \;\; \sqrt{\Delta x^2 + \Delta y_i^2} \;\; = \;\; \sqrt{\Delta x^2 + [f'(x_i^*)\Delta x]^2} \;\; = \;\; \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

so

$$\mathbf{L} \;\; = \;\; \lim_{n \to \infty} \left| P_{i-1} \; P_i \right| \;\; = \;\; \int_a^b \sqrt{1 + [f'(x)]^2} \; dx$$

Definition

If f' is continuous on $[a, b]$ then the length of the curve $y = f(x)$, $a \le x \le b$, is:

$$\mathbf{L} \;\; = \;\; \int_a^b \sqrt{1 + [f'(x)]^2} \; dx$$

or

$$\mathbf{L} \;\; = \;\; \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2} \; dx$$

ex 8 Find the length of the curve on $1 \le x \le 3$

$$y = \frac{2}{3}\left(x^2 - 1\right)^{3/2}$$

So

$$y' = (x^2 - 1)^{1/2}(2x) \implies 1 + (y')^2 = 1 + 4x^2(x^2 - 1) = 1 + 4x^4 - 4x^2 = (2x^2 - 1)^2$$

therefore

$$\mathbf{L} \ = \ \int_1^3 \sqrt{(2x^2-1)^2} \ dx \ = \ \int_1^3 (2x^2-1) \ dx \ = \ \ldots \ = \ \frac{46}{3}$$

Worksheet for Section 4

1. Find the length of $x = \frac{1}{3}\sqrt{y}(y-3), \qquad 1 \leq y \leq 9$

   HINT: Notice that the function is expressed as $x =$ stuff in y.

Homework for Section 4

1. SET UP ONLY an integral for the length of $y = cos\ x$ from $0 \leq x \leq 2\pi$
2. Find the length of $y = 1 + 6x^{3/2}$, $0 \leq x \leq 1$

5 Area of a Surface of Revolution

A *surface of revolution* is formed when a curve is rotated about a line.

We will focus on what is called the *lateral boundary* of the solid, **NOT** the bases. That is, not the top and bottom.

As always, first we will approximate with a basic shape and then take a limit.

What shape shall we use to approximate?

How did we approximate the length of the curve? We used line segments.

If one of these segments is rotated about an axis, what will the shape be?

Essentially you get the bottom portion of a circular cone called a *frustum*

The area of this band ends up being $A = 2\pi r l$ where r is the average radius of the band.

Each band has radius $f(x)$ and the width is the length of the line segment. Thus the area is approximated by

$$\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\,\Delta x$$

The surface area depends on the rotation

Surface Area is:

1. If we rotate the curve about the x-axis, then the circumference is $2\pi y$ and

$$\mathbf{S} = \int 2\pi y \; ds$$

2. If we rotate the curve about the y-axis, then the circumference is $2\pi x$ and

$$\mathbf{S} = \int 2\pi x \; ds$$

where

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx$$

or

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \; dy$$

Which one depends on how the curve is described. That is:

$$y = f(x) \qquad a \leq x \leq b \tag{1}$$

or

$$x = g(y) \qquad c \leq y \leq d \tag{2}$$

This allows us to calculate surface area without the graph being crucial to the set-up.

ex 9 We will do this first example both ways, that is with both (1) and (2). Let the arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ be rotated about the y-axis. Find the surface area

Since the rotation is about the y axis we know

$$S = \int 2\pi x \; ds$$

FIRST (1)

Let

$$y = x^2 \quad \Longrightarrow \quad \frac{dy}{dx} = 2x$$

So

$$S = \int 2\pi x \; ds \;\; =$$

$$\int_1^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \;\; dx$$

Note that the bounds are based on **x** *since we are integrating with respect to x*

$$= 2\pi \int_1^2 x\sqrt{1+4x^2}\ dx$$

This is then a substitution problem with $u = 1+4x^2$ and $du = 8xdx$

$$= \frac{2\pi}{8} \int_1^2 u^{1/2}\ du \ = \ \ldots \ =$$

$$\frac{\pi}{6}\left(1+4x^2\right)^{3/2} \Big|_1^2 = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

SECOND (2)

Let

$$x = \sqrt{y} \quad \Longrightarrow \quad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

So

$$S = \int 2\pi x\ ds \ = \ \int_1^4 2\pi\sqrt{y}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

Note that the bounds are now based on **y** *since we are integrating with respect to y*

$$= 2\pi \int_1^4 \sqrt{y}\sqrt{1+\frac{1}{4y}}\ dy \ =$$

$$\pi \int_1^4 \sqrt{4y+1}\ dy$$

Note that

$$\sqrt{y}\sqrt{1+\frac{1}{4y}} = \sqrt{y+\frac{y}{4y}} =$$
$$\sqrt{\frac{4y+1}{4}} = \frac{1}{2}\sqrt{1+4y}$$

This is then a substitution problem with $u = 4y + 1$ and $du = 4dy$

$$= \frac{\pi}{4}\int_1^4 \sqrt{u}\ du = \dots =$$
$$\frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^4 =$$
$$\frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$

ex 10 Find the area of the surface obtained by rotating $y = \sqrt{x}$ from $4 \le x \le 9$ about the x-axis.

Since the rotation is about the x axis we know

$$S = \int 2\pi y\ ds$$

Let

$$y = \sqrt{x} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

So

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

Therefore

$$S \; = \; \int_4^9 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \; dx \; =$$

$$\int_4^9 2\pi\sqrt{x}\sqrt{1+\left(\frac{1}{4x}\right)} \; dx$$

$$= 2\pi \; \int_4^9 \sqrt{x+\frac{1}{4}} \; dx \; =$$

$$2\pi \left[\frac{2}{3}\left(x+\frac{1}{4}\right)^{3/2} \; |_4^9\right] =$$

$$\frac{\pi}{6}\left(37\sqrt{37}-17\sqrt{17}\right)$$

Worksheet for Section 5

1. Find the area of the surface obtained by rotating the the curve $y = \sqrt[3]{x}, \quad 1 \le y \le 2$ about the y-axis.
2. Find the area of the surface obtained by rotating the the curve $x = 1 + 2y^2, \quad 1 \le y \le 2$ about the x-axis.

Homework for Section 5

1. SET UP ONLY an integral for the area of the surface created by rotating $y = x^4$ from $0 \leq x \leq 1$ about the x-axis AND the y-axis
2. Find the area of the surface by rotating $y = x^3\ ,\ \ 0 \leq x \leq 2$ about the x-axis.