1 Counting Part 1

ex 1

Let's look at todays lunch menu:

APPETIZERS

chicken tenders

salad

MAINS

tuna melt

burger

grilled cheese

DRINKS

iced tea

coffee

milkshake

gatorade

Some questions:

1. How many different lunches are there consisting of one appetizer, one main and one drink?
2. How about one main and one drink?
3. How about one main and an *optional* beverage?

These are counting problems. These kinds of questions can be answered using two basic principles.

MULTIPLICATION PRINCIPLE

If an activity can be constructed in t steps, step 1 in n_1 ways, step 2 in n_2 ways, ..., step t in n_t ways, then the number of possible activities is:

$$n_1 \cdot n_2 \cdot \quad \ldots \quad \cdot n_t$$

ex 2

How many 8 bit strings, each bit is a 1 or a 0, begin either 101 or 111?

Starting with 101, we essentially have a 5 step process:

step 1, select the fourth bit
step 2, select the fifth bit
$\vdots$
step 5, select the eighth bit

Since there are two choices at each step, the multiplication principle says that there are $2^5 = 32$ ways.

Starting with 111, the same process applies. So what's the total?

ADDITION PRINCIPLE

Suppose $X_1, X_2, \ldots, X_t$ are sets and the i^{th} set, X_i, has n_i elements. If $X_1, X_2, \ldots, X_t$ are pairwise disjoint, then the number of elements selected from X_1 or X_2 or ... or X_t is:

$$n_1 + n_2 + \ \ldots \ + n_t$$

so, from **ex 2**, the total would be $32 + 32 = 64$

PIGEONHOLE PRINCIPLE

If $k+1$ pigeons are placed in k pigeonholes, then at least one pigeonhole contains 2 or more pigeons.

This is also referred to as the Shoe Box Principle or the Dirichlet Drawer Principle.

Note that this is merely an existence argument, it does NOT identify which hole contains more than one pigeon.

ex 3

Show that if we select 151 distinct math courses numbered 1 to 300 inclusive, then at least 2 courses are consecutively numbered.

Let's look at two lists of distinct numbers.

$$\begin{aligned} \text{let the course numbers be} \quad & c_1, c_2, \ldots, c_{151} \\ \text{now let's also look at} \quad & c_1 + 1, c_2 + 1, \ldots, c_{151} + 1 \end{aligned}$$

Do you agree that the numbers on *each list* are distinct from each other?

Why did I create the second list by adding 1? Why not add 2 or 3 to each value from the first list?

Now, how many numbers are represented by the following list?

$$c_1, c_2, \ldots, c_{151}, c_1 + 1, c_2 + 1, \ldots, c_{151} + 1$$

The list above has 302 items, they range in value from 1 to 301, why?

$c_1, c_2, \ldots, c_{151}$ are <u>distinct</u> AND $c_1 + 1, c_2 + 1, \ldots, c_{151} + 1$ are <u>distinct</u>

$\Longrightarrow$?

$$c_i = c_j + 1 \text{ for some } i, j$$

PERMUTATIONS AND COMBINATIONS

Definition 1

A *permutation* of n distinct elements, $x_1, x_2, \ldots, x_n$ is an ordering of the n elements.

ex 4

How many permutations are there of the letters ABC? What are they?

Theorem 1

There are $n!$ permutations of n elements

Definition 2

A r permutation of n distinct elements, $x_1, x_2, \ldots, x_n$ is an ordering of an r element subset of $\{x_1, x_2, \ldots, x_n\}$. It is denoted $P(n, r)$.

ex 5

Find all of the 2 permutations of a, b, c. What are they?

Theorem 2

$$P(n, r) = \frac{n!}{(n - r)!}$$

ex 6

How many ways can we select a chair, vice chair, secretary and treasurer from a group of 10 people?

Really I'm just asking you to find what?

$$P(10, 4) = \frac{10!}{(10 - 4)!} = 5040$$

Definition 3

A selection of objects without regard to order is called a *combination*.

Theorem 3

The number of r combinations of a set of n distinct objects is

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

ex 7

How many ways can we select a four person committee from a group of 10 people?

$$C(10,4) = \frac{10!}{(10-4)!4!} = 210$$

Worksheet for Section 1

1. Using the letters $ABCDE$:

   (a) How many strings of length 4 can be formed with no repetitions?

   (b) How many strings from part (a) begin with B?

   (c) How many strings from (a) do not begin with B?

2. Use the multiplication principle to show that a set with n elements has 2^n subsets.

3. In how many ways can we select 2 books from different subjects among 5 distinct computer science books, 3 distinct math books and 2 distinct art books?

4. A six person committee composed of A,B,C,D,E and F is to select a chairperson, a secretary and a treasurer:

   (a) In how many ways can this be done?

   (b) In how many ways can this be done if either A or B must be chairperson?

   (c) In how many ways can this be done if E must hold one of the offices?

   (d) In how many ways can this be done if both D and F must hold an office?

2 Counting Part 2

Probability

Definition 4

A *sample space* consists of all possible outcomes. The probability of an event, denoted $P(E)$, is given by

$$P(E) \quad = \quad \frac{\text{the number of times the event occurs}}{\text{number of outcomes in the sample space}}$$

ex 8

If two fair dice are rolled (we will discuss what it means for dice to be fair), find the probability that the sum is ten.

How many possible outcomes are there?

How many add to ten?

$$P(E) = \frac{3}{36} = \frac{1}{12}$$

ex 9

Find the probability that among n persons, at least two people have have birthdays on the same date (not year). We shall ignore Feb29th.

This example is a great way to illustrate that sometimes it is easier to find the probability of an event not taking place.

If an event is certain, then $P(E) = 1$. If an event is impossible, then $P(E) = 0$. Therefore,

$$P(E) + P(\overline{E}) = 1$$

So
$P(E) =$ at least two people have the same birthday
$P(\overline{E}) =$ no two people have the same birthday.

Which is easier to find? Note that $P(E) = 1 - P(\overline{E})$.

The sample space is 365^n. For $P(\overline{E})$:
For the first person's birthday there are 365 choices
For the second person's birthday there are 364 choices
For the third person's birthday there are 363 choices

$$\vdots$$

For the n^{th} person's birthday there are 365 - n+1 choices

$$P(E) = 1 - P(\overline{E}) = 1 - \frac{365 \cdot 364 \cdot \ldots \cdot (365 - n + 1)}{365^n}$$

For $n = 22$, the probability is $\approx .476$
For $n = 23$, the probability is $\approx .507$

ex 10

The infamous Monty Hall problem. There are three doors in front of you. Behind one, a new car, behind the other two a goat. The idea is to pick the door with the car. Here's the question, you pick a door, Monty then shows you what's behind one of the two remaining doors (containing a goat of course), should you switch or stay with your original choice?

We will discuss the initial probability versus switching in class.

Worksheet for Section 2

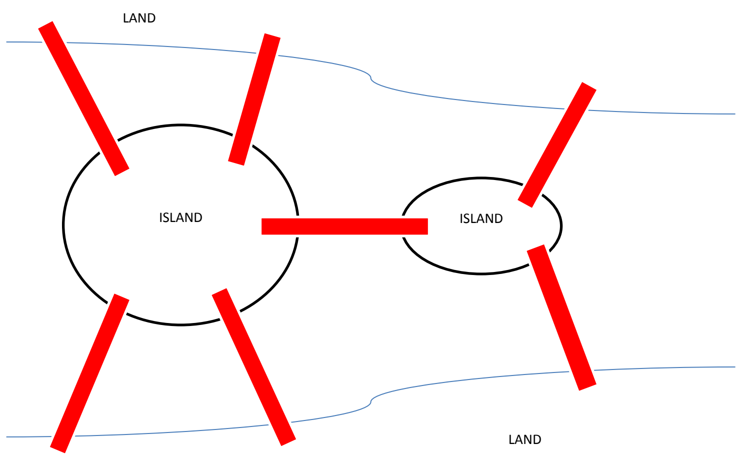
1. Using the letters $ABCDEF$:
   (a) How many permutations contain the substring DEF?
   (b) How many permutations contain the substring DEF in any order?
2. In how many ways can 7 women and 5 men wait in line if no 2 men stand together?
3. In how many ways can we select a committee of three from a group of ten people?
4. In how many ways can we select a committee of 2 women and 3 men from a group of 5 women and six men?
5. A deck of cards has 4 suits, clubs, diamonds, hearts and spades, and 13 denominations, ace, 2-10, jack, queen and king:
   (a) How many unordered 5 card poker hands are there?
   (b) How many poker hands contain cards of the same suit?
   (c) How many poker hands contain 3 cards of one denomination and 2 cards of a second denomination?

3 Graph Theory Part 1

Let's introduce the field of graph theory with an example.

In 1736 the residents of Konigsberg enjoyed walking over the multiple bridges in town. They consulted with Leonard Euler, perhaps the greatest mathematician ever, on whether or not it was possible to start and stop at the same place and cross every bridge exactly once. His solution gave rise to graph theory.

A map is provided below. Can it be done?



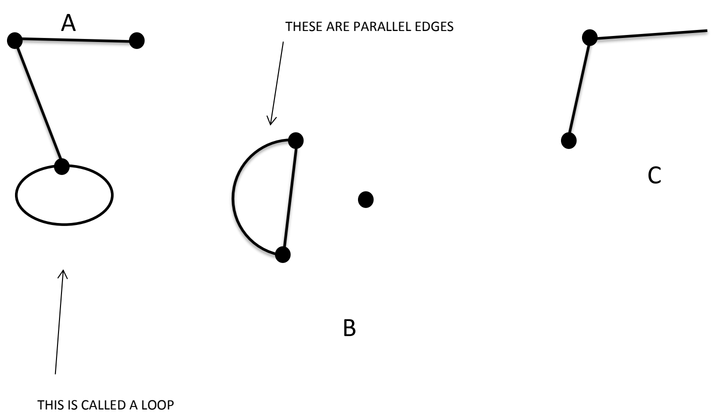
A graph is essentially a picture with dots (vertices) and lines (edges). More formally:

Definition 5

A graph G consists of a set of vertices V and a set of edges E such that each edge, $e \in E$ is associated with a pair of vertices.

Two vertices are *adjacent* if there is an edge between them.

Given the following pictures, A,B and C, which ones are graphs?

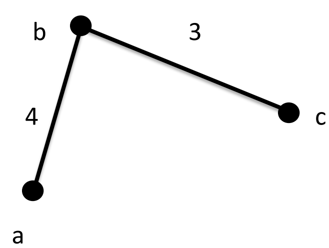


A graph with no loops or parallel edges is called a *simple graph*

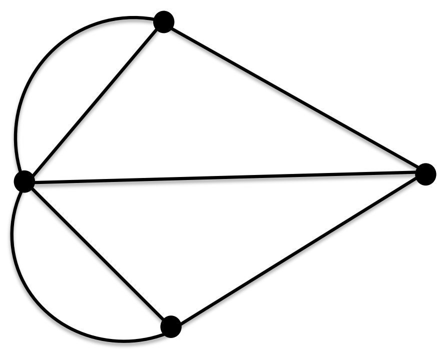
A graph with numbers on the edges is called a weighted graph. The length of the path is the sum of the weights.

ex 11

The path from a to c has what length?

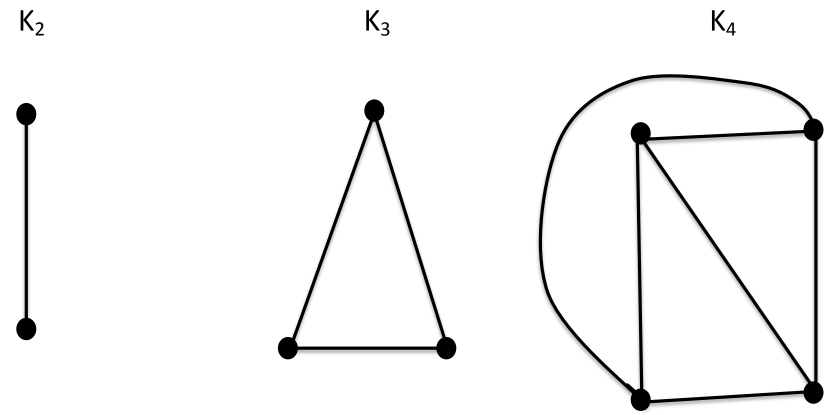


Now lets make a graph of the Konigsberg Bridge Problem...



Definition 6

The *complete* graph on n vertices, denoted K_n, is a simple graph with n vertices in which there is an edge between every pair of distinct vertices.



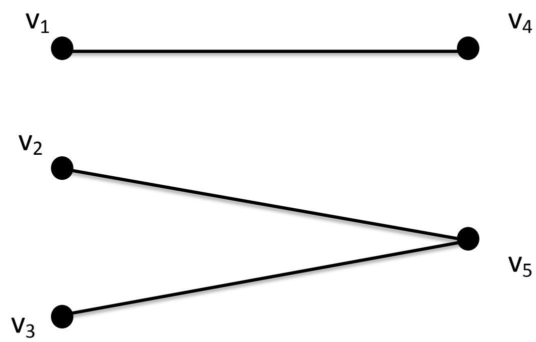
It looks like the degree of each vertex in K_n is what?

Definition 7

A graph, $G = (V, E)$, is *bipartite* if there exists subsets V_1 and V_2 of V, either possibly empty, such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$ and each edge in E is incident to one vertex in V_1 and one vertex in V_2.

ex 12

$V_1=\{v_1,v_2,v_3\}$ and $V_2=\{v_4,v_5\}$

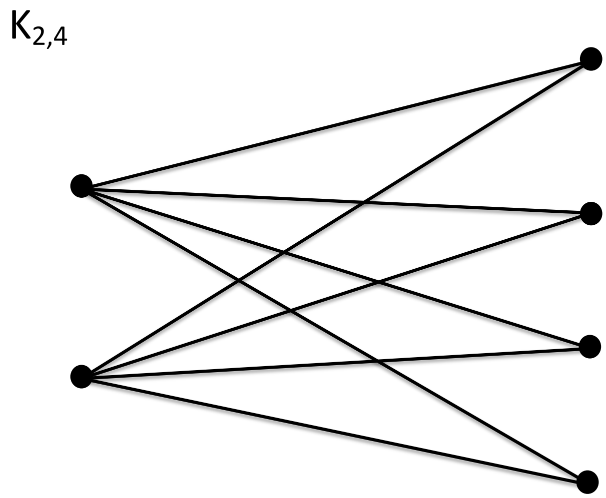


Note that if there exists an edge, $e \in E$, then it is incident to one vertex in V_1 and one vertex in V_2. It does not say that there must be an edge between them.

Definition 8

The *complete bipartite* graph on m and n vertices, denoted $K_{m,n}$, is the simple graph whose vertex set is partitioned into V_1 with m vertices and V_2 with n vertices such that there is an edge between each pair of vertices v_1 and v_2 where $v_1 \in V_1$ and $v_2 \in V_2$

ex 13



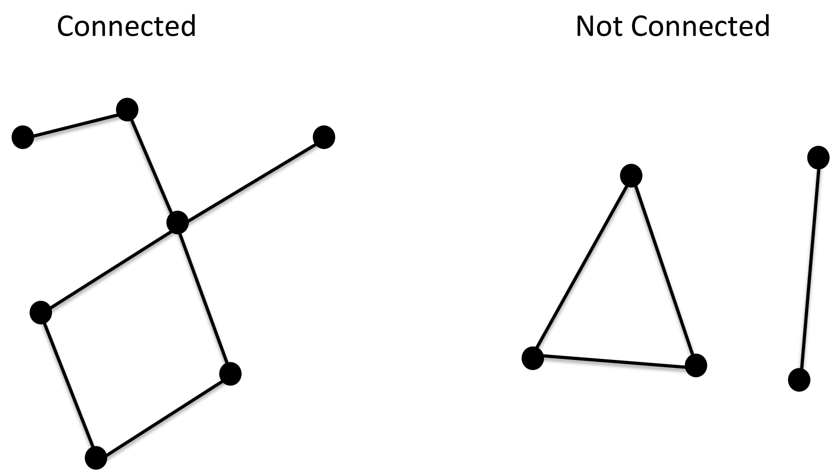
Definition 9

Let v_i and v_j be vertices in a graph. A *path* from v_i and v_j is an alternating sequence of vertices and edges beginning with v_i and ending with v_j.

Definition 10

A *connected graph* is a graph G such that given any vertices $v_i, v_j \in V$ there exists a path from v_i to v_j

ex 14

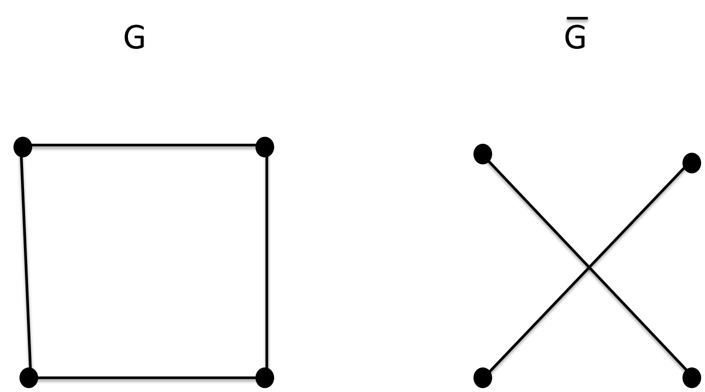


A connected graph consists of one *piece* while a graph that is not connected has two or more *pieces*. These *pieces* are subgraphs of the original and are called *components*.

Definition 11

The *complement* of a graph G, denoted $\overline{G}$, has the same vertices as G and if v_1, v_2 have an edge incident in G, they do not in $\overline{G}$.

ex 15



Theorem 4

For any graph G, either G or $\overline{G}$ is connected.

proof:

If G is connected, then we are done. So, let's assume G is not connected. This means that G has components, $C_1, C_2, \ldots, C_k$ such that each component is connected but no edge exists between any C_i and C_j. We shall show that $\overline{G}$ is connected.

case 1

Let $x, y \in V(G)$ with x and y being in different components of G. Then by definition, there exists an edge between x and y in $\overline{G}$

case 2

Let $x, y \in V(G)$ with x and y in the same C_i of G. So, $x, y \in C_i$. Let z be some vertex in C_j such that $C_i \neq C_j$. If there is no edge between x and y in G, then there is in $\overline{G}$. If there is an edge between x and y in G then in $\overline{G}$ there exists an edge from x to z AND from y to z. This implies that x, y and z are linked by a path. Since x, y and z are arbitrary, if G is not connected, then $\overline{G}$ is.

■

Definition 12

Let $v, w \in V(G)$. A *cycle* is a path from v to v with no repeated edges.

An *Euler cycle* is a cycle that includes all of the edges and all of the vertices of G.

The Konigsberg Bridge problem is an attempt to find an Euler cycle.

Theorem 5

A graph G has an Euler cycle $\iff$ G is connected and every vertex has even degree.

Theorem 6

A graph G has a path with no repeated edges from v to w, $v \neq w$, containing all of the edges and vertices $\iff$ G is connected and v and w are the only vertices of odd degree.

Worksheet for Section 3

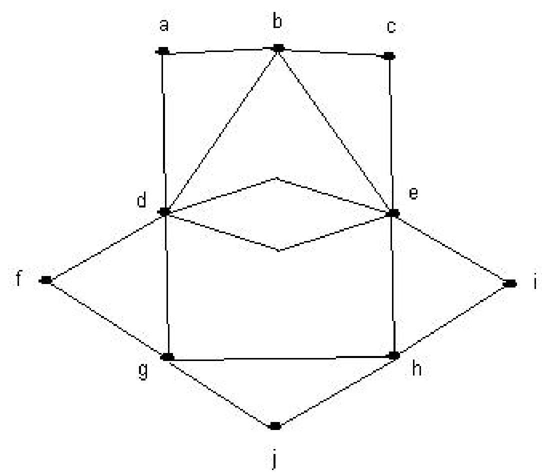
1. Draw a graph with the given properties or explain why none exists.

   (a) 6 vertices each degree 3

   (b) 6 vertices and 4 edges

   (c) A simple graph with 6 vertices having degrees 1,2,3,4,5,5

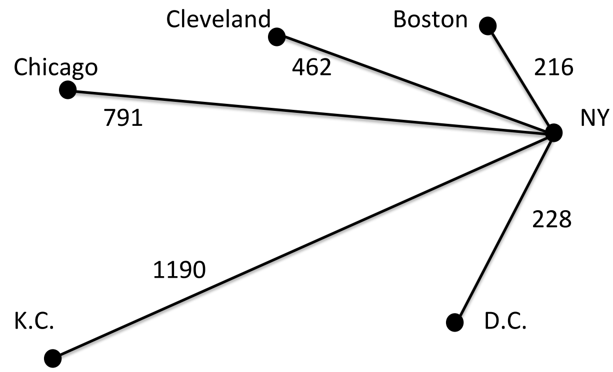
2. Given the following graph, is there an Euler cycle? If so find one.



3. Let G be a graph. Define a relation R on the set V of vertices of G as vRw if there is a path from v to w. Prove R is an equivalence relation on V.

4 Graph Theory Part 2

THE TRAVELING SALESMAN PROBLEM



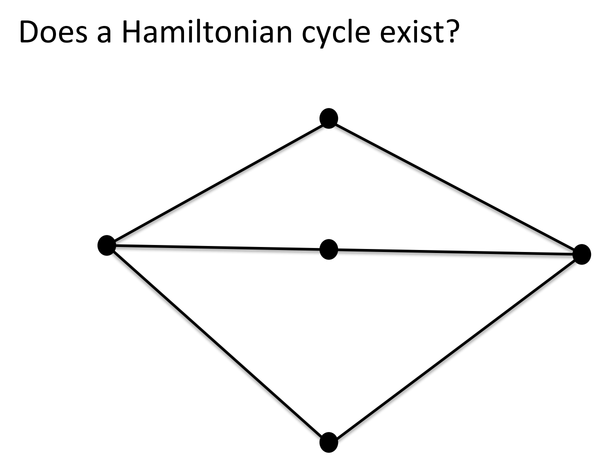
Using the mileage chart below, visit each city exactly once, except where you start, in the fewest miles.

	NY	Boston	D.C.	Cleveland	Chicago	K.C.
NY	-	216	228	462	791	1190
Boston	216	-	441	639	984	1431
D.C.	228	441	-	371	701	1074
Cleveland	462	639	371	-	346	798
Chicago	791	984	701	346	-	526
K.C.	1190	1431	1074	798	526	-

Definition 13

A cycle in a graph G that contains each vertex, except the starting vertex, exactly once is called a *Hamiltonian Cycle*.

ex 16



Unfortunately the only way to determine the answer to the traveling salesman problem is by *brute force*, that is, check every possible combination. As this is extremely inefficient, there are some approximation methods.

CHEAPEST LINK METHOD

First, choose the shortest path. Then, continue with the next shortest path as long as a vertex does not reach degree 3. Continue until you have a cycle.

ex 17

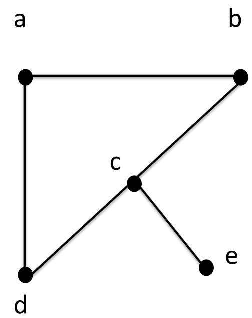
From our original problem...

First, choose NY to Boston, 216 miles
next, NY to D.C., 228 miles
then Cleveland to Chicago at 346
next, D.C. to Cleveland at 371
next is Boston to D.C. at 441 (why can't we choose this?)
then 462, NY to Cleveland (also a no go)
then 526, Chicago to KC
next 639, Boston to Cleveland (a no go)
$\vdots$
finally, we need KC to Boston at 1431.

REPRESENTATIONS OF GRAPHS

One such representation is an ***adjacency matrix***

- select an ordering of the vertices
- label the rows and columns with the ordered vertices
- the entry in row i, column j, $i \neq j$, is the number of edges incident on i and j
- if $i = j$, the entry is twice the number of loops



So,

$$\mathbf{A} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

In a simple graph G, $A, A^2, A^3, \ldots$ provides the number of paths of varying lengths.

ex 18

$$\mathbf{A^2} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 2 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 \\ 2 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

In row a, column c, the entry is 2. This means that there are 2 paths of length 2 from a to c. Verify this.

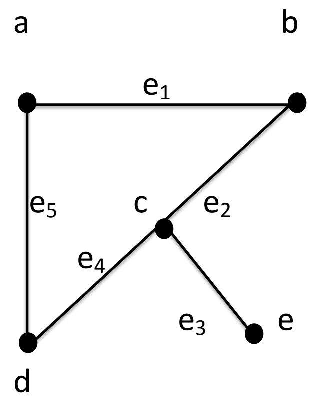
ex 19

$$\mathbf{A^4} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 8 & 0 & 10 & 0 & 0 \\ 0 & 9 & 0 & 9 & 5 \\ 10 & 0 & 13 & 0 & 0 \\ 0 & 9 & 0 & 9 & 5 \\ 0 & 5 & 0 & 5 & 3 \end{pmatrix} \end{array}$$

In row d, column e, the entry is 5. This means that there are 5 paths of length 4 from d to e. Verify this.

Another such representation is an ***incidence matrix***

- label the rows with vertices and the columns with edges.
- the entry for row v, column e is 1 if e is incident to v and 0 otherwise.



$$\mathbf{B} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

* Note that the sum of the row is the degree of the vertex.

Definition 14

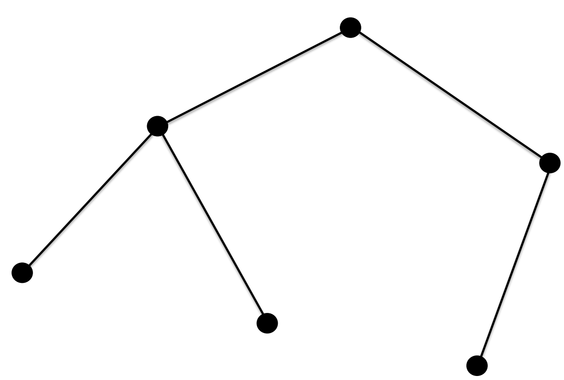
A *forest* is a graph with no cycles.

Definition 15

A *tree* is a graph with no cycles that is connected, that is, a connected forest.

The vertices of degree one are sometimes referred to as leaves.

ex 20



Definition 16

A *bridge* is an edge e in a connected graph G such that $G - e$ is disconnected.

Theorem 7

If T is a graph with n vertices, the following are equivalent.

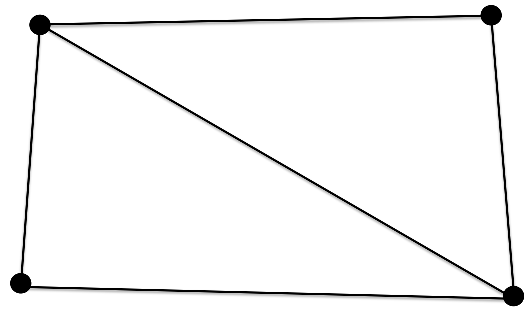
1. T is a tree.
2. T has no cycles and $n - 1$ edges.
3. T is connected and has $n - 1$ edges.
4. T is connected and each edge is a bridge.
5. The addition of any edge creates a cycle.
6. Any two vertices are connected by exactly one path.

Definition 17

A tree T is a *spanning tree* of a graph G if T is a subgraph of G that contains all of the vertices of G.

Why do you think spanning trees are so practical?

ex 21



Find all of the spanning trees of the graph G

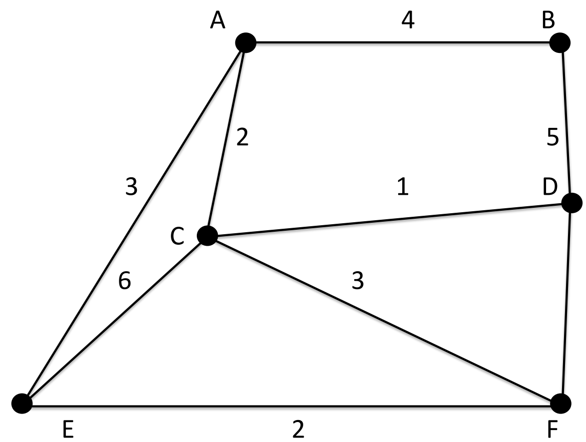
Theorem 8

A graph G has a spanning tree $\iff$ G is connected.

Definition 18

If G is a weighted graph, a *minimal spanning tree* of G is a spanning tree of G with the minimum weight.

ex 22



We will use KRUSKAL'S ALGORITHM to find minimal spanning trees.

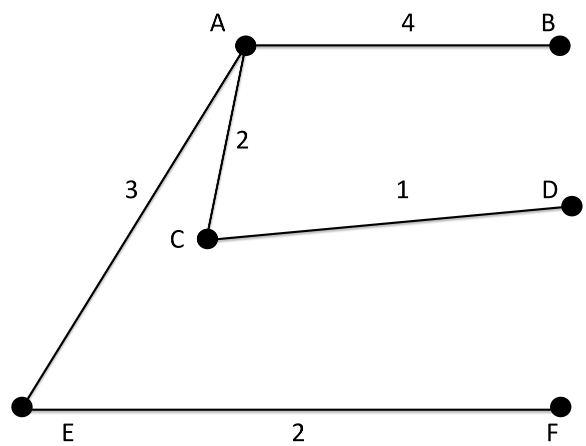
• start with all of the vertices and no edges.

• at each step (or iteration) add the edge of **minimum** weight that does <u>not</u> complete a cycle.

• when T has $n-1$ edges, stop.

ex 23

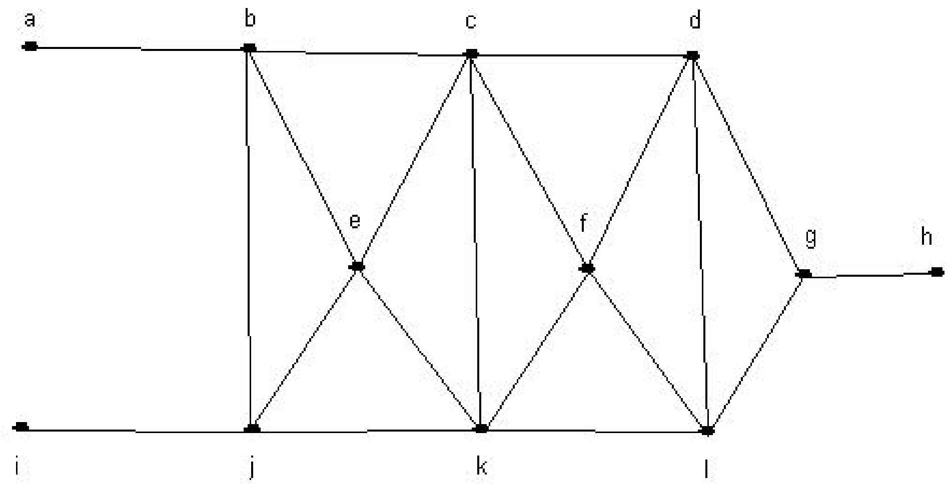
Let's find the minimal spanning tree from the previous example.



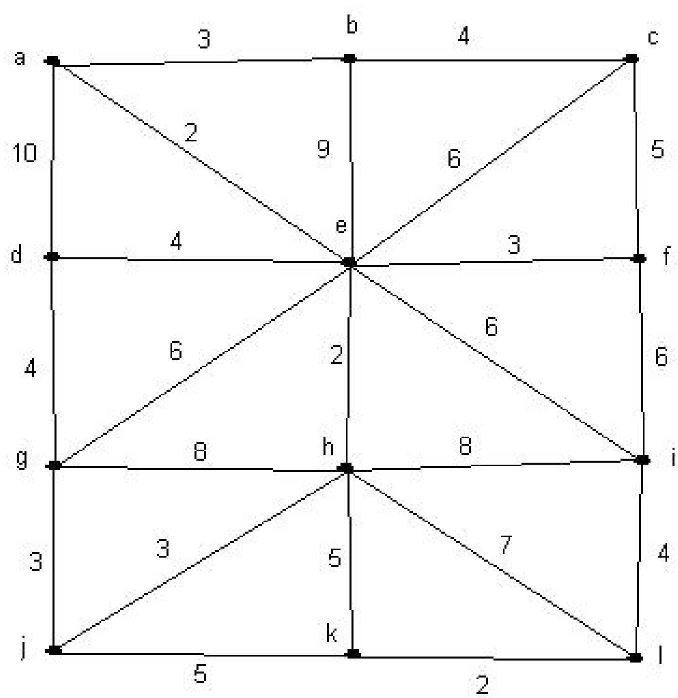
Here the minimal spanning tree has weight 12.

Worksheet for Section 4

1. Find a spanning tree of the following graph.



2. Find a minimal spanning tree of the following graph.

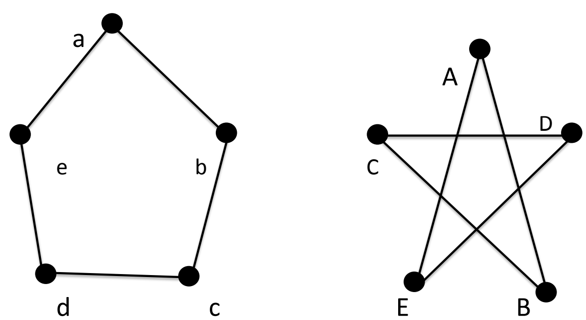


5 Graph Theory Part 3

ISOMORPHISMS OF GRAPHS

Let's try the following exercise, draw and label 5 vertices, a, b, c, d and e. Now connect the following, a to b, b to c, c to d, d to e and a to e.

What does your graph look like?



Which is correct?

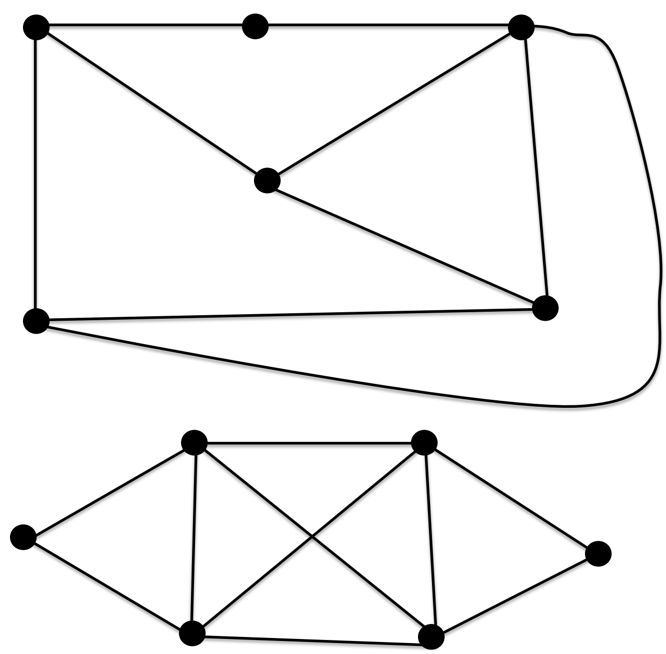
Definition 19

Two graphs, G_1 and G_2, are *isomorphic* if there exists a bijection $f : V(G_1) \mapsto V(G_2)$ and a bijection $g : E(G_1) \mapsto E(G_2)$ so that e is incident to v and w in $G_1 \iff g(e)$ is incident to $f(v)$ and $f(w)$ in G_2.

Theorem 9

G_1 and G_2, are isomorphic $\iff$ for some ordering of their vertices their adjacency matrices are equal.

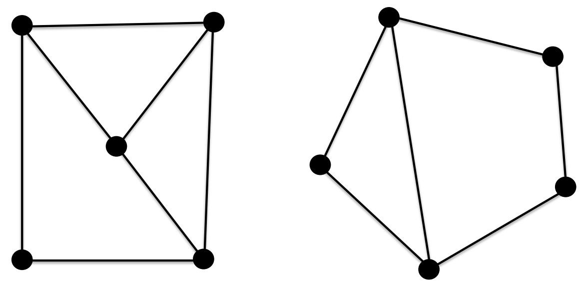
ex 24



Are these isomorphic?

Why or why not?

ex 25



Are these isomorphic?

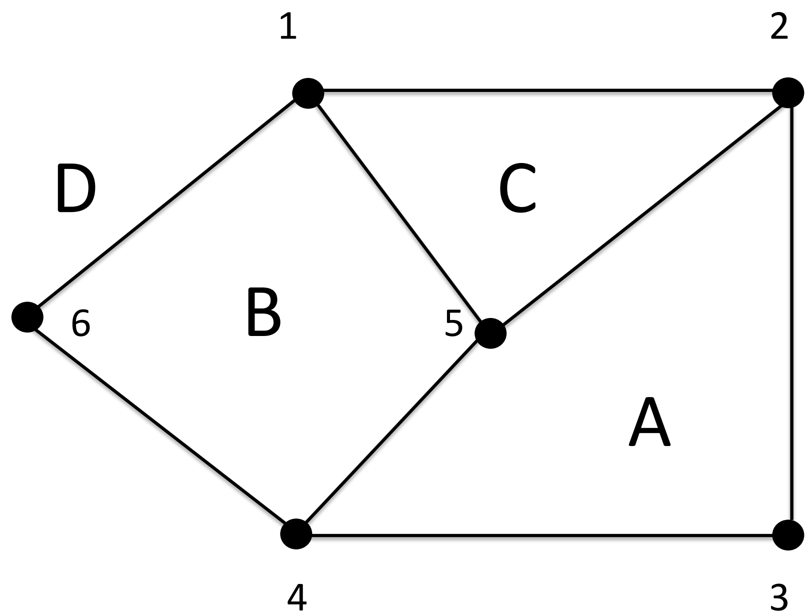
Why or why not?

Definition 20

A graph is *planar* if it can be drawn in the plane **without** its edges crossing.

If a connected planar graph is drawn, it is divided into regions called *faces* which are characterized by the cycle that forms its boundary.

ex 26



Face A is bounded by (5,2,3,4,5)

Face B is bounded by (1,5,4,6,1)

Face C is bounded by (1,2,5,1)

Face D is bounded by (1,2,3,4,6,1)

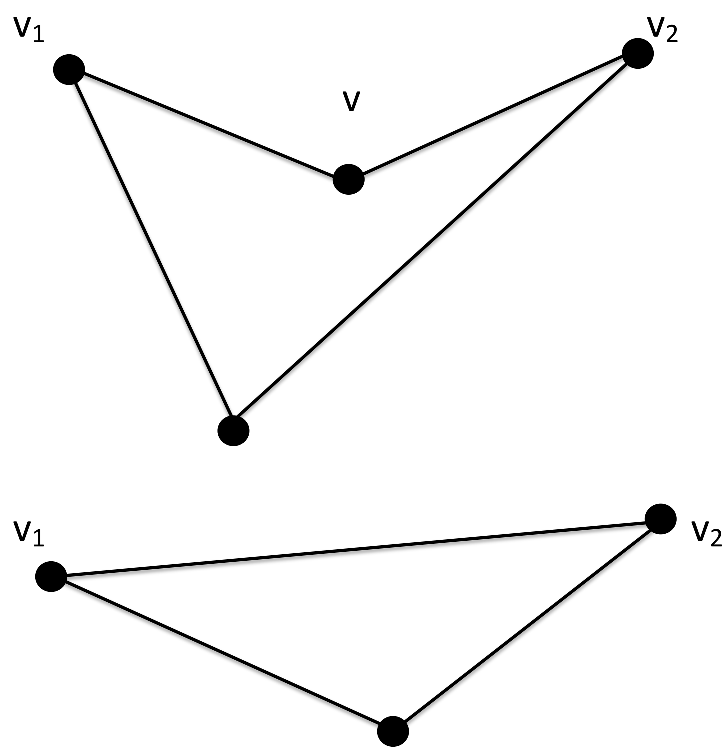
How many faces are there? How many edges? Vertices?

Note that

$$f = e - v + 2$$

Euler proved that this formula holds true for every connected planar graph.

ex 27

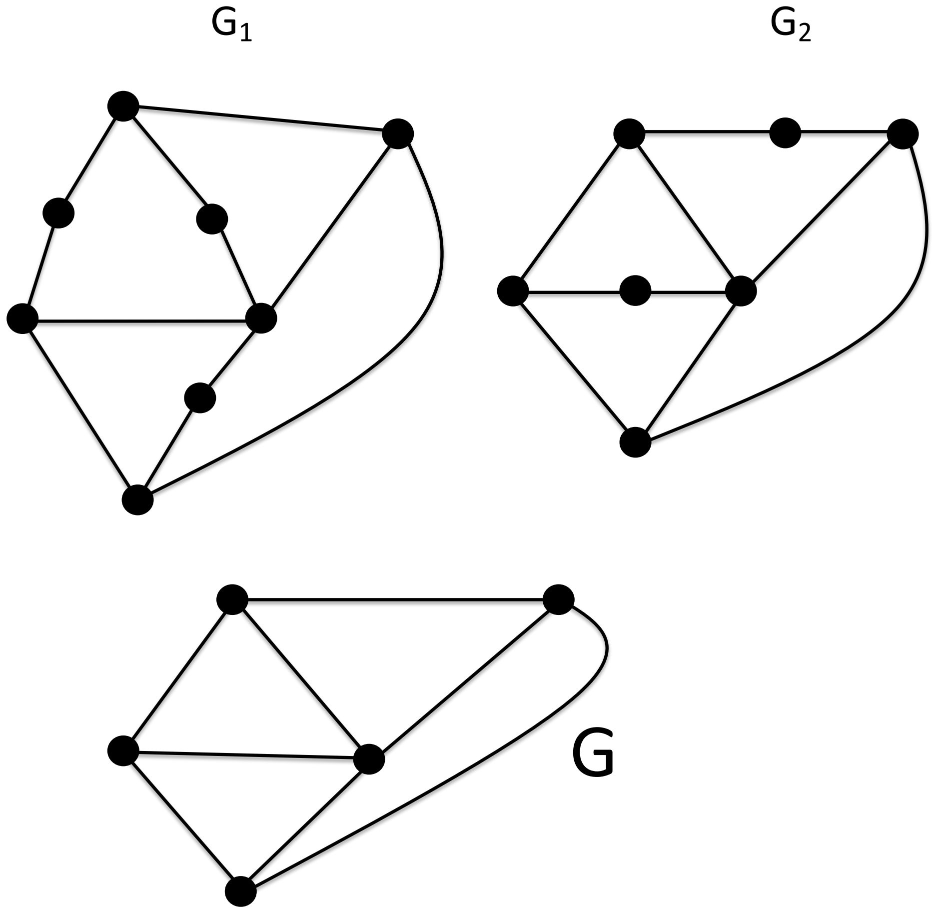


In the top graph, (v_1,v) and (v,v_2) are in series. The bottom graph is obtained from what is called a <u>series reduction</u>.

Definition 21

Two graphs, G_1 and G_2, are *homeomorphic* if G_1 and G_2 can be reduced to isomorphic graphs by performing a sequence of series reductions.

ex 28



G_1 and G_2 are homeomorphic since they can both be reduced to G

KURATOWSKI'S THEOREM

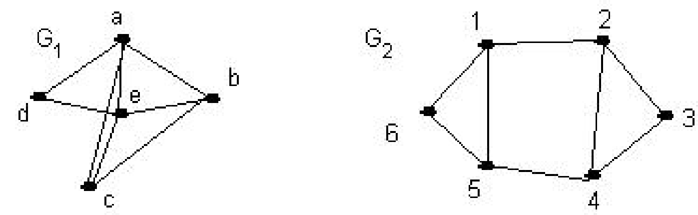
A graph G is planar $\iff$ G does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$

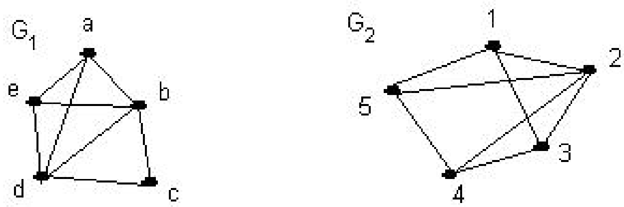
ex 29

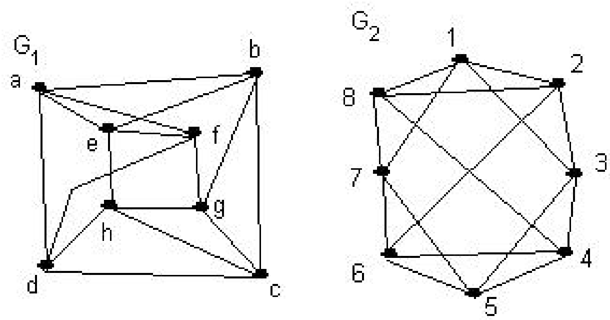
see worksheet

Worksheet for Section 5

1. For each pair determine if G_1 and G_2 are isomorphic.







2. Show that G is NOT planar. Try to find $K_{3,3}$